Topological Hochschild and cyclic homology for K-theory

Sergio Maciel

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Topics

- Act I: classical Hochschild and Cyclic homology
- 2 Act II: Topological Hochschild homology
- 3 Act III: Action
- 4 Act IV: K-theory

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$$d_j(a_0, a_1, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_j \cdot a_{j+1}, \ldots, a_n) & \text{if } j < n; \\ (a_n \cdot a_0, a_1, \ldots, a_{n-1}) & \text{if } j = n. \end{cases}$$

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The first character we will meet is the Hochschild homology of A, obtained as the homology of the Hochschild complex and it is denoted by $HH_{\bullet}(A)$.



The cyclic action

Observe the existence of an action of the cyclic group C_n on each component of the Hochschild complex. Denote by t the generator of the cyclic group C_n and we have

$$t(a_0,\ldots,a_{n-1})=(a_{n-1},a_0,\ldots,a_{n-2}).$$

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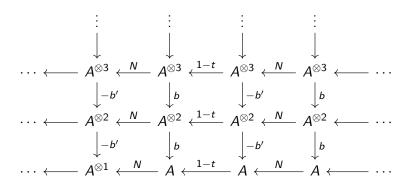
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This "enriches" the Hochschild complex allowing for the definition of the cyclic homology.

Classicaly, we first define the map $N = 1 - t + t^2 - \cdots + t^n$ and together with the morphism b', which is the boundary of the bar complex of A, we build the following bicomplex.

The cyclic bicomplex and cyclic homology



The cyclic homology is then the total homology of this bicomplex. This is a central character on our story, which we denote by $HC_{\bullet}(A)$.

Delving into our cast I: the Hochschild homology

The Hochschild complex is actually a *Moore complex*. The d_i maps we used to define the boundary operator turn the Hochschild complex into a *simplicial module*. The hochschild complex is the associated complex with this simplicial module under the Dold-Kan correspondence.

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Consequently, the geometric realization of this simplicial module yields a space whose homotopy type represents the Hochschild complex:

$$\pi_n\Big(\big|\cdots\to A\otimes A\otimes A\to A\otimes A\to A\big|\Big)\cong \mathsf{HH}_n(A).$$



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More sophisticated tools allow us to pin down how exactly this action is "responsible" for the existence of a cyclic homology.

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To any ring R, we can associate its Eilenberg-Mac Lane spectrum HR, which has components $HR_n = K(R, n)$. This construction is functorial and embedds **CRing** into the category of \mathbb{S} -algebras.



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$$THH(A) := \Big| \cdots \rightrightarrows HA \wedge_{\mathbb{S}} HA \rightrightarrows HA \Big|.$$

Leading character: topological Hochschild homology

So far, we straight up imitated Hochschild homology. But, instead of ending up with a space, we got a spectrum. In the same vein the Hochschild homology is represented by the mentioned space (via its homotopy groups), we shall define the topological Hochschild homology to be represented by the Hochschild spectrum.

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Similarly as before, THH(A) also has a circle action, which we can use to define topological cyclic homology, but we first need to make sense of that statement.



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In 2017, Thomas Nikolaus and Peter Scholze had the insight that one can use this more relaxed notion of equivariant spectra instead, provided you put it in the appropriated language.

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Restating what we have commented earlier: THH(A) is a spectrum with an S^1 -action.

Homotopy fixed points and homotopy orbits

When a group G acts on a ring, we can look out for fixed points and orbits. It is no different here, but we are dealing with homotopy-sensitive algebraic objects, so we will account for that.

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$$X_{hG} = \text{hocolim } X.$$

The Tate construction

We apply a Tate construction to homotopy orbits and fixed points. Recall that Tate cohomology can be defined as the cofiber of the norm map from group homology to group cohomology. Analogously, we define the $Tate\ spectrum\ of\ X$ as the (homotopy) cofiber

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We will be interested in the cases $G=S^1$ or $G=C_p$. If one mimics what we described here but swaping spectra for chain complexes (or more precisely, the derived category of a ring), we recove classical cyclic homology: $HC(A) \cong HH(A)_{hS^1}$, while $HH(A)^{hS^1}$ and $HH(A)^{tS^1}$ correspond to negative and periodic cyclic homology.

The rise: cyclotomic spectra

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$$\phi_p: X \to X^{tC_p}$$

spectrum with an S^1 action X together with S^1 -equivariant maps

for each prime number p.



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We define the topological cyclic homology of a cyclotomic spectrum \boldsymbol{X} as the coequalizer

$$TC(X) o X^{hS^1} \stackrel{\operatorname{can}}{\underset{\prod_{\rho} \phi_{\rho}}{\Longrightarrow}} \prod_{\rho} (X^{tC_{\rho}})^{hS^1}.$$

THH(A) is **always** a cyclotomic spectrum.

For any ring A, we denote by TC(A) the spectrum TC(THH(A)).



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Now, lets move to our topological enrichment.



As before, we have some kind of trace map from K(A) to TC(A), called the *cyclotomic trace*.

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- applications for assembly maps;

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• $\mathsf{K}_{2n-1}(k[C_{p^N}]; \mathbb{Z}_p) \cong \mathsf{K}_1(k[C_{p^N}]; \mathbb{Z}_p)^{\oplus n}$ and $\mathsf{K}_{2n}(k[C_{p^N}]; \mathbb{Z}_p) = 0$ for k of characteristics p;

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- extending the result above, we also know the K-theory of truncated polynomial algebras [Angeltveit, Gerhardt, Hill, Lindenstrauss].

What is the point?

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The upshot is that TC and THH are MUCH MUCH more easy to compute than K-theory. There are lots of spectral sequences available (K-theory is not so generous).

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The upshot is that TC and THH are MUCH MUCH more easy to compute than K-theory. There are lots of spectral sequences available (K-theory is not so generous).

There are other uses of TC and THH in algebra besides K-theory. For instance, we have the work of Bhatt, Morrow and Scholze in *p*-adic cohomology.

References

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