Topological Hochschild and cyclic homology for K-theory

Sergio Maciel

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Topics

1. Act I: classical Hochschild and Cyclic homology
2. Act II: Topological Hochschild homology
3. Act III: Action
4. Act IV: K-theory

Topics

Enters: the Hochschild complex

Given a k-algebra A, we form its Hochschild complex

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$$d_j(a_0, a_1, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_j \cdot a_{j+1}, \ldots, a_n) \text{ if } j < n; \\ (a_n \cdot a_0, a_1, \ldots, a_{n-1}) \text{ if } j = n. \end{cases}$$

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The first character we will meet is the Hochschild homology of A, obtained as the homology of the Hochschild complex and it is denoted by $\mathrm{HH}_\bullet(A)$.

The cyclic action

Observe the existence of an action of the cyclic group C_n on each component of the Hochschild complex. Denote by t the generator of the cyclic group C_n and we have

$$t(a_0, \dots, a_{n-1}) = (a_{n-1}, a_0, \dots, a_{n-2}).$$

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This "enriches" the Hochschild complex allowing for the definition of the cyclic homology.
Classicaly, we first define the map $N = 1 - t + t^2 - \cdots + t^n$ and together with the morphism b', which is the boundary of the bar complex of A, we build the following bicomplex.

The cyclic bicomplex and cyclic homology

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
\cdots & \longleftarrow & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
\cdots & \longleftarrow & A^{\otimes 1} & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \longleftarrow & \cdots
\end{array}
$$

The cyclic homology is then the total homology of this bicomplex. This is a central character on our story, which we denote by $\mathrm{HC}_\bullet(A)$.

Delving into our cast I: the Hochschild homology

The Hochschild complex is actually a *Moore complex*. The d_i maps we used to define the boundary operator turn the Hochschild complex into a *simplicial module*. The hochschild complex is the associated complex with this simplicial module under the Dold-Kan correspondence.

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Consequently, the geometric realization of this simplicial module yields a space whose homotopy type represents the Hochschild complex:

$$\pi_n\Big(\Big|\cdots \to A\otimes A\otimes A \to A\otimes A \to A\Big|\Big) \cong \mathrm{HH}_n(A).$$

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There are multiple senses in which the C_n actions are incarnations of S^1-homotopy theory. Maybe the simplest way to put that is that the geometric realization of cyclic objects come with a canonical action of the circle.
More sophisticated tools allow us to pin down how exactly this action is "responsible" for the existence of a cyclic homology.

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Moving to spectra

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To any ring R, we can associate its Eilenberg-Mac Lane spectrum HR, which has components $HR_n = K(R, n)$. This construction is functorial and embedds **CRing** into the category of $\mathbb{S}$-algebras.

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$$THH(A) := \left| \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} HA \wedge_{\mathbb{S}} HA \rightrightarrows HA \right|.$$

Leading character: topological Hochschild homology

So far, we straight up imitated Hochschild homology. But, instead of ending up with a space, we got a spectrum. In the same vein the Hochschild homology is represented by the mentioned space (via its homotopy groups), we shall define the topological Hochschild homology to be represented by the Hochschild spectrum.

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Similarly as before, $\mathrm{THH}(A)$ also has a circle action, which we can use to define topological cyclic homology, but we first need to make sense of that statement.

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We define a *spectrum with a G-action* to be a functor $\mathbf{B}G \to \mathbf{Sp}$.
This is a more naive notion of equivariant spectra than the ones people have been using in equivariant stable homotopy theory. In fact, everything we are doing from now on was first figured out for these more complicated notions.
In 2017, Thomas Nikolaus and Peter Scholze had the insight that one can use this more relaxed notion of equivariant spectra instead, provided you put it in the appropriated language.

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Restating what we have commented earlier: $\mathrm{THH}(A)$ is a spectrum with an S^1-action.

Homotopy fixed points and homotopy orbits

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and the homotopy orbits as

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The Tate construction

We apply a Tate construction to homotopy orbits and fixed points. Recall that Tate cohomology can be defined as the cofiber of the norm map from group homology to group cohomology.
Analogously, we define the *Tate spectrum* of X as the (homotopy) cofiber

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We will be interested in the cases $G = S^1$ or $G = C_p$. If one mimics what we described here but swaping spectra for chain complexes (or more precisely, the derived category of a ring), we recove classical cyclic homology: $\mathrm{HC}(A) \cong \mathrm{HH}(A)_{hS^1}$, while $\mathrm{HH}(A)^{hS^1}$ and $\mathrm{HH}(A)^{tS^1}$ correspond to negative and periodic cyclic homology.

The rise: cyclotomic spectra

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If X is a spectrum with an S^1 action, it also has a C_p action induced by the inclusion $C_p \subset S^1$. Similarly, X^{tC_p} has an S^1 action as a residue of the quotient $S^1 \cong S^1/C_p$.

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Hence, it makes sense to define a *cyclotomic spectrum* as a spectrum with an S^1 action X together with S^1-equivariant maps

$$\phi_p : X \to X^{tC_p}$$

for each prime number p.

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We define the *topological cyclic homology* of a cyclotomic spectrum X as the coequalizer

$$TC(X) \to X^{hS^1} \underset{\prod_p \phi_p}{\overset{\text{can}}{\rightrightarrows}} \prod_p (X^{tC_p})^{hS^1}.$$

THH(A) is **always** a cyclotomic spectrum.

For any ring A, we denote by TC(A) the spectrum TC(THH(A)).

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Now, lets move to our topological enrichment.

From K-theory to topological cyclic homology

As before, we have some kind of trace map from $\mathrm{K}(A)$ to $\mathrm{TC}(A)$, called the *cyclotomic trace*.

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- applications for assembly maps;

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- $\mathrm{K}_{2m-1}(k[x]/(x^n);\mathbb{Z}_p) = W_{nm-1}(k)/V_n W_{m-1}(k)$ for k perfect of characteristic p, where $W(k)$ is the Witt ring and $V_n(f(x)) = f(x^n)$;

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- extending the result above, we also know the K-theory of truncated polynomial algebras [Angeltveit, Gerhardt, Hill, Lindenstrauss].

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The upshot is that TC and THH are MUCH MUCH more easy to compute than K-theory. There are lots of spectral sequences available (K-theory is not so generous).
There are other uses of TC and THH in algebra besides K-theory. For instance, we have the work of Bhatt, Morrow and Scholze in p-adic cohomology.

References

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