The stable and unstable homotopy theory of spectra

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December 2023

Introduction

The Suspension Theorem due to Freudenthal gave birth to stable homotopy theory. Mathematicians were originally interested on how topological spaces behaved when iterating the suspension functor. The idea of spectra appeared to study in more details the Spanier-Whitehead duality. While trying to understand stability properties of the suspension functor, it became more and more evident that looking at some object that contained all the data of successive iterations of the suspension functor could be a good idea.

Thereof was born the concept of spectra, a sequence of topological spaces with maps connecting the suspension of the last to the next one. Quickly after the debut of spectra, notions of their stable homotopy groups were thrown around and some form of homotopy theory for spectra was born. The thoughts of doing homotopy theory with things that are not topological spaces really got into the spotlight after Quillen's model categories arrived in this world. After that, mathematicians quickly layed down the fundamentals of the homotopy theory of spectra in terms of this new language that model categories provided.

Meanwhile, the study of "homotopy coherent algebra" was full steam and many ideas of how to formalize such a thing started popping up. Between these many proposals of what should be the definition of certain homotopy coherent algebraic objects (most latter found to be loosely equivalent), Segal's Γ-spaces received some attention, in particular by Bousfield and Friedlander. In their 78 paper [BF78], trying to build a model structure for Γ-spaces, they described one of the earliest examples of Bousfield localizations (didn't have this name yet) in order to explain how one of the model structures for spectra could be used to create another one, which at some point was understood to be the right. One year later, Bousfield published a paper showing a more systematic way to do a very similar procedure, which originated the term *Bousfield Localization*.

After these years of development, we had these two model structures describing two different homotopy theories of spectra: the *strict model structure for spectra* and the *stable model structure for spectra*. While the strict model was based on a perspective of spectra merely as sequences of spaces with some extra structure (so one could try some componentwise approach), the stable model structure tried to grasp some behaviors observed in the realm of spectra strongly related to the stability properties of the suspension functor, the fundamental tool for defining spectra. More than simply the existence of two different model structures for spectra, the stable one could be obtained by Bousfield Localizing the strict one.

Later on, more progress was made in different directions. In special, we may cite the recognition of spectra as some kind of diagram in the category of topological spaces. This allowed mathematicians to use some tropes from the theory of model categories to construct the strict model structure on spectra. Advancements in the theory of model categories with extra structure also gave us the blessing of stable model categories and the stabilization of model structures, which when applied to the strict model structure on spectra, results in the stable one.

Furthermore, the hunt for a nice category of spectra ended up bringing a lot of different special types of spectra. In order to make the category of spectra a symmetric monoidal one or even closed monoidal and transfer these structures to the relevant model structures, things like symmetric, orthogonal spectra and similar new flavors of spectra needed to be introduced.

This is an exposition on the two main different model structures in the category of spectra: the strict and the stable one. We will discuss what they are and how they are related. The first section is designed for either the reader who needs a little reminder on the very foundational aspects of stable homotopy theory or the ones who are not familiar with the subject and want to see some cool examples of homotopy theories.

The second section describes the strict model structure on the category of spectra. We work on some details about cofibrant replacement and CW-approximation for this model structure. We take a different approach than that taken by Bousfield and Friedlander in [BF78], for example. They work with the corresponding notion of spectrum in the category of simplicial sets. We prefer to stick directly with the topological incarnation as the classical homotopy theoretical results and concepts we will use are in general better disseminated.

Third section concerns Bousfield localization of model categories. There are many possible takes one can have about Bousfield localizations, varying in level of technicality. Perhaps the most useful one is via local weak equivalences, but we definitely don't need to engage with the technicalities inherited by such approach. So we opt for a much easier-to-digest definition with the downside that it forces one to work more to obtain precise clarifying information about the things we are working with. In practice, they are equivalent.

The last section introduces the stable model category of spectra. It uses some results explained in the last section to clarify the process of "stabilization" (we don't actually define what *stabilization* is) of the strict model structure from the second section. We point out some basic features of this model structure and finish with some comments about stable model categories, from which the stable model category of spectra is a prototypical example, and how all of this relates to the more contemporary point of view regarding homotopy coherent mathematics.

1 Preliminars on spectra

This section is dedicated to reviewing the fundamental notions we will be working with. If the reader is familiar with the concept of *spectra* and the basic constructions associated with it, this may be skipped without any loss.

The idea of spectra is to organize a list of topological spaces in a sequence where the next space is connected to the suspension of the previous. One may wonder why are spectra even a thing, that is, why people care to define such kind of object. There are many reasons for mathematicians to pay attention to them. A first reason may be the fact that they rise naturally when considering homotopy groups of subsequent suspensions of spaces.

Maybe an even better reason is that spectra are naturally related to cohomology theories. For instance, spectra represent generalized cohomology theories, in the sense that every generalized cohomology theory has an associated spectrum that allows computing the cohomology of a space via maps into such spectrum.

First of all, we need to define what is the suspension of a space. We will work with pointed spaces, so our definitions will be "pointed-sensible". The idea of the suspension is to build a new topological space ΣX from an old one X by attaching two "cones" from above and below X.

Definition 1 (Reduced suspension)**.** *The* ***reduced suspension*** ΣX *of a pointed topological space* $(X, *)$ *is the space*

$$\frac{X \times [0,1]}{(X \times \{1\}) \cup (X \times \{0\}) \cup (\{*\} \times [0,1])}$$

The picture to have in mind is that ΣX is built by stacking many copies of X, forming a cylinder $X \times [0,1]$ with base X and then collapsing the ends. In addition, each copy of X used in the cylinder has a distinguished point given by the basepoint of X. If one marks all such distinguished points $(*, t)$ for each $t \in [0,1]$, the result will be a line crossing the cylinder from one end to another. We also collapse this whole line, pulling the two collapsed ends of the cylinder to the same point. Then, this point is taken to be the base point of ΣX.

Moreover, if $f : (X, x_0) \to (Y, y_0)$ is a map of pointed topological spaces, we can build a map between the suspensions $\Sigma f : \Sigma X \to \Sigma Y$ by sending the equivalence class of (x, t) to $(f(x), t)$, so we just apply f in each level of the cylinder. One can quickly check this map is well-defined. Furthermore, it is straightforward to verify the relations $\Sigma(f \circ g) = \Sigma f \circ \Sigma g$ and $\Sigma\, \mathrm{id}_X = \mathrm{id}_{\Sigma X}$. Hence, Σ forms a *functor* $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ from the category of pointed topological spaces to itself.

There is another way to describe the suspension functor, which is via another operation called *smash product*. The coproduct in the category $\mathbf{Top}_*$ of pointed topological spaces is the wedge sum, defined below.

Definition 2 (Wedge sum)**.** *The* ***wedge sum*** *of two pointed topological spaces* (X, x_0) *and* (Y, y_0) *is defined to be the space* $X \vee Y$ *given by*

$$\frac{X \sqcup Y}{x_0 \sim y_0}.$$

Which is pointed by the equivalence class $[x_0] = [y_0]$.

This construction basically takes two spaces and glues them along the base-points. It defines a bifunctor as two maps $f : X \to Y$ and $g : X' \to Y'$ induce a map $f \vee g : X \vee X' \to Y \vee Y'$ that receives a point in $X \vee X'$, choose the function which is defined in the corresponding component of the wedge sum (that is, X or X') and maps the point to $Y \vee Y'$ using such function. We finally have the smash product.

Definition 3 (Smash product)**.** *Let* (X, x_0) *and* (Y, y_0) *be two pointed topological spaces. We define the* ***smash product*** $X \wedge Y$ *to be the space*

$$\frac{X \times Y}{(X \times \{y_0\}) \cup (\{x_0\} \times Y)}.$$

pointed by the point corresponding to the collapsed subspace.

Notice that the subspace we quotient by in the smash product is homeomorphic to a wedge sum $X \vee Y$, so we could write $X \wedge Y = X \times Y / X \vee Y$ with the proper identification.

Again, $\wedge$ defines a bifunctor in $\mathbf{Top}_*$. If we fix one of the inputs to be S^1, we obtain a functor $S^1 \wedge (-) : \mathbf{Top}_* \to \mathbf{Top}_*$ which coincides with the suspension functor Σ we defined previously. This description is interesting because $\wedge$ acts like a tensor product in pointed spaces. More precisely, $\wedge$ turns the category of compactly generated pointed spaces into a closed symmetric monoidal category. In this situation, currying gives an isomorphism

$$\mathrm{Map}(X \wedge Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

.

By setting $Y = S^1$, we recover a version of the loop space-suspension adjunction. Another possible statement of, essentially, the same result is the following.

Theorem 4. *For any pointed spaces* X, Y*, there is a bijection*

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

between homotopy classes of maps from ΣX *to* Y *and homotopy classes of maps from* X *to the loop space of* Y*.*

The mentioned bijection is given by associating a function $f : \Sigma X \to Y$ to the function $h(x) = f(x, -)$. For a fixed x, the function $f(x, -)$ indeed defines a loop as f is a function over the suspension of X and the points $(x, t) \in \Sigma$ with $t = 0$ or $t = 1$ are identified.

Now, we finally define what a spectrum is.

Definition 5 (Spectrum)**.** *A **spectrum** is a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed topological spaces and pointed maps $\Sigma X_n \to X_{n+1}$.*

Example 1 (Eilenberg-MacLane spectrum)**.** *Given an abelian group G and a natural number n, one can form the space $K(G,n)$, which is a CW complex characterized by the property that*

$$\pi_i(K(G,n)) = \begin{cases} G \text{ if } i = n, \\ 0 \text{ otherwise.} \end{cases}$$

There is a homotopy equivalence between $\omega K(G, n+1)$ and $K(G,n)$, yielding, via the loop space-suspension adjunction, a map $\Sigma K(G,n) \to K(G,n+1)$ that turns the sequence of spaces $K(G,n)$ into a spectrum.

*This spectrum is called the **Eilenberg-Maclane spectrum** of G and is denoted by HG.*

Example 2 (Sphere spectrum)**.** *Computations show that ΣS^n is homeomorphic to S^{n+1}, so we have a spectrum $\mathbb{S}$, called the **sphere spectrum** whose n-th element is the sphere S^n. The maps in this spectrum are the homeomorphisms $\Sigma S^n \cong S^{n+1}$.*

Example 3 (Suspension spectrum of a space)**.** *Given a space X, it is possible to form the **suspension spectrum** of X, denoted by $\Sigma^\infty X$, whose n-th element is $\Sigma^n X$, the space obtained by applying the suspension functor n times to X. The maps of this spectrum are the identity maps.*

With the terminology above, the sphere spectrum can be identified with the suspension spectrum of the two-point space S^0. To do this properly, we need to understand what a map of spectra is.

Definition 6. *A **map between spectra** $f : X = (X_n)_n \to Y = (Y_n)_n$ is a sequence of pointed maps $f_n : X_n \to Y_n$ making the following diagram commute*

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array} .$$

The vertical maps are the structure maps of the spectra.

Then, an **isomorphism** is a morphism of spectra which is a homeomorphism levelwise.

Our final goal is to describe the homotopy theory of spectra in terms of model categories. Knowing what kind of thing we are trying to describe in terms of model categories certainly will help, so we finish this preliminary section by describing very grounded versions of homotopy groups of spectra and homotopies between maps of spectra.

First of all, notice that there is a canonical map $\pi_k(X_n) \to \pi_{k+1}(X_{n+1})$ for all k, n. As $\pi_k(X_n) = [S^k, X_n]$, applying Σ to each map $S^k \to X_n$ yields a map $[\Sigma S^k, \Sigma X_n] = [S^{k+1}, \Sigma X_n]$, which further maps to $[S^{k+1}, \Sigma X_n]$ via the structure map of X. We used that Σ preserves homotopy classes. Thus, the following definition makes sense.

Definition 7 (Homotopy groups of spectra)**.** *Given a spectrum X, its **k-th homotopy group** is defined to be the colimit*

$$\operatorname{colim}(\cdots \to \pi_{k+n}(X_n) \to \pi_{k+n+1}(X_{n+1}) \to \cdots).$$

As for topological spaces, maps $X \to Y$ between spectra induce maps in its homotopy groups, obtained as the colimit of the induced maps in the topological spaces defining X and Y. One can expect already a definition of weak-equivalence of spectra as a map that induces isomorphisms in all homotopy groups of spectra.

Constructions from topological spaces can be extended to spectra. For example, we can define the wedge sum of a pointed topological space with a spectrum, and also a smash product. These are done componentwise and, being bifunctors, we have induced maps between the new spaces, also defining a spectrum. Notice that, with this language, the suspension spectrum of X is precisely $X \wedge \mathbb{S}$, the smash of X with the sphere spectrum.

As a consequence, there are functors $\Sigma, \Omega : \mathbf{Sp} \to \mathbf{Sp}$ from the category of spectra to itself given by applying the suspension and loop space functor componentwise. These are also adjoints.

2 The strict homotopy theory of spectra

Spectra are just sequences of topological spaces. It is true that they have a bit more structure than any sequence, but they are still sequences, just special ones. As such, we can approach a homotopy theory for spectra in a very naive way, doing everything componentwise. To formalize this idea in terms of model structures, we first need a category. Luckily, we already have everything for that. We let **Sp** be the category of spectra, whose objects are, well... Spectra, and morphisms are maps of spectra, as in Definition 6. This is the first step towards a homotopy theory of spectra. The next ones are a bit more involving.

2.1 Defining the strict model structure

We said we were going to do everything componentwise. What it means is that our special classes of morphisms for our model category of spectra (weak equivalences, fibrations and cofibrations) will be defined componentwise, as follows.

Definition 8. *A map of spectra $f : X \to Y$ is said to be*

1. ***a weak equivalence** if each $f_n : X_n \to Y_n$ is a weak equivalence in the Quillen model structure for topological spaces;*

2. ***fibration** if each $f_n : X_n \to Y_n$ is a fibration in the Quillen model structure for topological spaces;*

3. ***cofibration*** *if it has the left lifting property with respect to trivial cofibrations.*

It is hinted already by the choice of names that these classes of maps will turn **Sp** into a model category, which we call the **strict model structure on spectra**. While this is the case, this is not the "right" model category, since it doesn't account for stability phenomena observed in spectra, as we will explain later. But we can build the right model structure from this one.

Showing that this actually gives a model structure is not that simple. (Co)completeness of **Sp** and the 2-out-of-3 property are fairly easy, as well as closure under retracts. But lifting properties and factorizations, although happening componentwise, have no apparent reason for defining maps of spectra. We need to do a small detour.

The more experienced reader may have noticed the similarity of the way we defined our distinguished classes of morphisms with the way one defines these classes for other situations. This "componentwise" idea is recurrent in many situations and the model structure is reminiscent of model structures in diagram categories.

In our case, we can look at model structures in categories of functors. The two main ones are the *projective* and *injective* model structures. We can see how spectra look like functors from, let us say, a category of natural numbers to pointed topological spaces. But they have a small twist arising from the structure maps, so this perspective is not quite right. But we can make it work if we put some effort into changing the way we see spectra as functors. The claim is that **Sp** is equivalent to a functor category. Which functor category?

Definition 9. *We denote by* **Spheres** *the category whose objects are the spheres* S^n *and hom sets are given by*

$$\operatorname{Hom}(S^n, S^{n+k}) = \begin{cases} * \text{ for } k < 0; \\ S^k \text{ otherwise.} \end{cases}$$

The functor category Fun(**Spheres**, $\mathbf{Top}_*$) is equivalent **Sp**. The equivalence takes a functor F : **Spheres** $\to$ $\mathbf{Top}_*$ and builds the spectrum whose n-th component is $F(S^n)$. The structure maps are declared using the loop space-suspension adjunction and the composition in **Spheres**, which we did not describe as we don't want to dive any deeper into the technicalities of these constructions. The upshot is that we can consider the projective model structure on Fun(**Spheres**, $\mathbf{Top}_*$), whose existence is guaranteed by the **Top**-enrichment of **Spheres**.

Now, we can transfer this model structure to **Sp** via the said equivalence. The result is what we declared as weak equivalences, fibrations and cofibrations before.

Another approach is to look at the category of spectra as a diagram category [Man+01], we end up with a fairly concrete description of the cofibrations (see *framing* in [Hov]):

A map $f : X \to Y$ is a cofibration if $f_0 : X_0 \to Y_0$ is a cofibration and all pushforward maps

$$X_{n+1} \sqcup_{\Sigma X_n} \Sigma Y_n \to Y_{n+1}$$

are also cofibrations.

Then we have some interesting facts about this model structure. The suspension spectrum construction defines a functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}$ and, as we now show, this functor has a right adjoint Ω^∞.

Definition 10. *We define the functor $\Omega^\infty : \mathbf{Sp} \to \mathbf{Top}_*$ by $\Omega^\infty(X) = X_0$, that is, it picks the 0-th component.*

Theorem 11. *There is an adjunction $\Sigma^\infty \dashv \Omega^\infty$.*

Proof. Given a spectrum X and Y a topological space, then $\mathrm{Hom}(\Sigma^\infty Y, X)$ is the set of sequences $f_n : \Sigma^n Y \to X_n$ such that the diagrams

$$\begin{array}{ccc} \Sigma\Sigma^{n-1}Y & \xrightarrow{\Sigma f_{n-1}} & \Sigma X_n \\ \wr\big\downarrow & & \big\downarrow \\ \Sigma^n Y & \xrightarrow{f_n} & \Sigma X_n \end{array}$$

commute. Suppose f_n is already given. Then, since the arrow $\Sigma\Sigma^{n-1}Y \to \Sigma^n Y$ is an isomorphism, there is only one possible f_{n+1} making the diagram commute, which is given by composing all the inverse of the isomorphism with the other maps. Hence, all morphisms in $\mathrm{Hom}(\Sigma^\infty Y, X)$ are determined by $f_0 : Y \to X_0$. Recalling that $X_0 = \Omega^\infty X$, we can see the bijection.

$$\mathrm{Hom}(\Sigma^\infty Y, X) \cong \mathrm{Hom}(Y, \Omega^\infty X)$$

. □

What is interesting is that this adjunction is a Quillen adjunction between the strict model structure on **Sp** and the Quillen model structure on $\mathbf{Top}_*$. The proof of this fact is as simple as observing that whenever we have a morphism of spectra $f : X \to Y$ and we apply Ω^∞, we end up with the morphism $f_0 : X_0 \to Y_0$ and the definition of the strict model structure implies that whenever $f : X \to Y$ is a weak equivalence (resp. fibration or cofibration), so does $f_0 : X_0 \to Y_0$.

Proposition 12. *The adjoint functors $\Sigma : \mathbf{Sp} \leftrightarrow \mathbf{Sp} : \Omega$ form a Quillen adjunction.*

Proof. It follows from the fact that $\Sigma : \mathbf{Top}_* \leftrightarrow \mathbf{Top}_* : \Omega$ is a Quillen adjunction and the model structure in spectra is componentwise. □

2.2 Cofibrant objects in the strict model structure

We have done a bunch of stuff already with the strict model structure on spectra and we know things tend to happen componentwise. So we could expect fibrant and cofibrant replacements to also be done componentwise. Let us see how it goes.

We will ignore fibrant objects for the time, since they will drastically change later when we localize the strict model structure. First, the initial object in **Sp** is the spectrum whose n-th component is a point $*$ for all n. Then, for $X \in \mathbf{Sp}$ cofibrant, we need $* \to X$ to be a cofibration, which means $* \to X_0$ is a cofibration. So our first requirement is that X_0 must be a cofibrant object in the Quillen model structure, so a retract of a cell complex.

Also, $\Sigma(*) \cong *$ by a quick calculation. Hence, the pushforward $* \sqcup_{\Sigma(*)} \Sigma X_n$ is simply ΣX_n. So we need the structure map $\Sigma X_n \to X_{n+1}$ to be a cofibration.

This way, we obtain the following theorem

Theorem 13. *A spectrum X is cofibrant in the strict model structure of* **Sp** *if and only if X_0 is cofibrant and the structure maps $\Sigma X_n \to X_{n+1}$ are cofibrations.*

We also have a "CW approximation" for spectra, which serves as a form of cofibrant replacement This CW approximation is by CW-spectra, whose definition is the following.

Definition 14 (CW-spectrum)**.** *A* ***CW-spectrum*** *is a spectrum X such that each X_n is a CW-complex and the structure maps $\Sigma X_n \to X_{n+1}$ are all celular inclusions.*

Theorem 15. *Let $X \in$* **Sp** *be a spectrum. Then there is some strict weak equivalence $\hat{X} \to X$ from a CW-spectrum $\hat{X}$ to X.*

Proof. By usual CW-approximation, we have a weak equivalence $X_0 \to \hat{X}_0$ in the 0-th component. Now we build the next maps inductively.

Suppose we have a sequence of weak equivalences $f_k : \hat{X}_k \to X_k$ for all $k \leq n$ such that the "structure diagrams" commute, that is, the diagrams

$$\begin{array}{ccc} \Sigma\hat{X}_k & \xrightarrow{\Sigma f_k} & \Sigma X_k \\ \downarrow & & \downarrow \\ \hat{X}_{k+1} & \xrightarrow{f_{k+1}} & X_{k+1} \end{array}$$

commute for all $k < n$.

Then, the upper path of such diagram when $k = n$ provides a map $\Sigma\hat{X}_n \to X_{n+1}$. CW approximation for this map gives a factorization

$$\begin{array}{ccc} \Sigma\hat{X}_n & \longrightarrow & X_{n+1} \\ & \searrow \quad \nearrow & \\ & Y_{n+1} & \end{array}$$

where Y_{n+1} is a CW-complex and the map $Y_{n+1} \to X_{n+1}$ is a weak equivalence. We set $\hat{X_{n+1}} = Y_{n+1}$ and this map the compatibility with the structure maps of X is a mere consequence of the commutativity of the last diagram. $\square$

Hence, after this bit of work, we know that there is a kind of cofibrant replacement in **Sp** that works just as CW approximation does in **Top**.

3 Bousfield localizations

In this section we will discuss Bousfield localizations, which is a type of localization we will use to "stabilize" our model structure on **Sp**. The general philosophy of a Bousfield localization is to mimic what one does when localizing ordinary categories at a class of morphisms, but in a homotopy coherent way. So we consider the same kind of diagram and final/cofinal problem we would when defining the localization of a category, but using Quillen functors instead of any functor.

A more pragmatic perspective is that of a solution to the problem of modifying a model structure in a controlled way. The "operation" of Bousfield localizing takes a model structure on a category and returns another model structure that has either the same set of fibrations or the same set of cofibrations, but in any case the number of weak equivalences should increase. *Why would one want to do such a thing?*

That is a valid question. Possible answers include: *I want to make more things weak-equivalent.* Or *I want to shorten my set of fibrant or cofibrant objects.* A further question that may appear is *why are these answers good answers?* For that question, one answer, perhaps not uniquely, is because sometimes we would love homotopy types to be stable under some kind of operation or maybe the interesting (co)fibrant objects are a subclass of the actual ones. To really appreciate these ideas, one would need to be exposed to some situations where these thoughts are the ones driving mathematicians to do certain things. Good examples include $\mathbb{A}^1$ homotopy theory, where Bousfield localizations appear to "make the affine line contractible" or in our case, nice spectra are ones in which the structure maps are weak equivalences. These are called Ω-spectra. Between the reasons why they are the interesting spectra, we have Brown's representability associating an Ω-spectrum to each cohomology theory and also, a more sophisticated reason is that Ω-spectra are the "actual spectra" in the category of spectra.

In short terms, every model category can be turned into a stable model category via the *stabilization* construction. The stabilization of a model category is naturally identified with the category of *spectrum objects* in the model category. The catch here is that spectrum objects are defined in the way they need to be so that the category of such objects is the stabilization of the model category. It turns out that the spectrum objects in the category of spectra are not all spectra, but the omega ones. We didn't discuss stable categories yet so this reasoning may feel a bit out of reach by now. But perhaps this foreshadowing of theory can help to motivate caring about Ω-spectra so much.

Definition 16 (Bousfield localization)**.** *Let $\mathcal{C}$ be a model category with weak equivalences W and cofibrations C. A **Bousfield localization** of $\mathcal{C}$ is a model category $\mathcal{C}_{loc}$ with the same underlying category and same cofibrations but its class of weak equivalences W_{loc} contains W.*

This is a pretty general definition of Bousfield localization. There are many ways to build Bousfield localizations for model categories. We are going to focus on the procedure that uses idemponent functors to do so, as our "intended" fibrant replacement functor will be one.

Theorem 17. ***Hypothesis:*** *Let $Q : \mathcal{C} \to \mathcal{C}$ be an endofunctor in a model category $\mathcal{C}$ and $\eta : \mathrm{id} \to Q$ a natural transformation satisfying:*

1. *Q preserves weak equivalences (also known as a homotopical functor);*
2. *for all $X \in \mathcal{C}$, $Q(\eta_X)$ and $\eta_{Q(X)}$ are weak equivalences;*
3. *if in a pullback square*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{h} & Z \end{array}$$

f is a fibration and the morphisms η_X, η_Y are weak equivalences, then $Q(P \to X)$ is also a weak equivalence.

Consequence: *The following classes determine a model structure in the underlying category $\mathcal{C}$:*

- *weak equivalences: maps f such that $Q(f)$ is a weak equivalence;*
- *cofibrations: maps that are a cofibration (they are the same);*
- *fibrations: maps having the right lifting property with respect to the trivial cofibrations as above.*

*These are called **Q-weak equivalences** and **Q-(co)fibrations**.*

This is a theorem due to Bousfield and Friedlander and the proof is a bit long, so we are going to skip it.

The important thing is that now we have a more or less algorithmic way to obtain more model structures in the same category. It may be at least interesting to see that these new model structures obtained this way are Bousfield localizations of the original one.

First of all, it should be clear that the class of weak equivalences increases. If f is a weak equivalence, then $Q(f)$ is also a weak equivalence as Q preserves this class of map. Then, since $Q(f)$ is a weak equivalence, f is, by definition, a Q-weak equivalence as well. Furthermore, the class of cofibrations is the same so, indeed, we obtain a Bousfield localization.

4 The stable homotopy theory of spectra

We will now Bousfield localize the strict model structure of spectra to obtain a model structure that is sensible to stabilization phenomena in spectra. What kind of phenomena are we talking about?

The most basic form may be the existence of stable homotopy groups. Spectra have homotopy groups, defined as certain colimits (see Definition 7) and that suggests the existence of a homotopy type based in these homotopy groups. But our current weak equivalences do not account for this kind of homotopy type, as they do not consider what happens in the colimit of the homotopy groups.

A hint about how to solve this problem comes from looking at the cases in which the sequence of homotopy groups stabilizes, so isomorphisms in each level will yield an isomorphism in the colimit as the sequence is constant.

In more details, let us say we have a map between spectra $f : X \to Y$ and we are looking at the induced map in the sequence of homotopy groups used in the colimit that computes $\pi_k(X)$ and $\pi_k(Y)$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{k+n}(X_n) & \longrightarrow & \pi_{k+n+1}(X_{n+1}) & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle (f_n)_*} & & \big\downarrow{\scriptstyle (f_{n+1})_*} & & \\ \cdots & \longrightarrow & \pi_{k+n}(Y_n) & \longrightarrow & \pi_{k+n+1}(Y_{n+1}) & \longrightarrow & \cdots \end{array}$$

Let us say the $(f_n)*$ maps are isomorphisms. How can we guarantee that the induced map on the colimit will be an isomorphism? Well... It happens if the sequence actually stabilizes, that is, becomes constant after a while. So we may now require isomorphisms $\pi_{k+n}(X_n) \cong \pi_{k+n+1}(X_{n+1})$. Thankfully, the loop space-suspension adjunction gives isomorphisms $\pi_k(X) \cong \pi_{k-1}(\Omega X)$, so the answers lies in Ω-spectra, as these are the ones whose sequence of homotopy groups become constant (maybe up to a shift on degree, and then we would expect Σ and Ω to induce equivalences on the homotopy category of **Sp** with this new model structure).

This is one of the motivations for us to go out in the woods and hunt for a funtor Q that turns spectra into Ω-spectra, which we would like to behave as some kind of fibrant replacement.

Definition 18 (Spectrification)**.** *We define a functor Q, called **spectrification** by the following construction.*

- *Start with a spectrum X and set $Z_0^k = X_k$;*
- *we are going to define a sequence $(Z_i^k)_{i\in\mathbb{N}}$ and maps $Z_i^k \to \Omega Z_i^{k+1}$ for each k inductively;*
- *assume we did for all $i < n$;*
- *take the map $Z_i^k \to Z_i^{k+1}$ and factors it as a cofibration followed by a weak equivalence $Z_i^k \to Y \to \Omega Z_i^{k+1}$;*

- *define* $Z^k_{i+1} = Y$*;*
- *do that for all* k*;*
- *the map* $Z^k_{i+1} \to \Omega Z^{k+1}_{i+1}$ *is given by applying* Ω *to the fibration* $Z^{k+1}_i \to Z^{k+1}_{i+1}$ *and composing the result with the weak equivalence* $Z^k_{i+1} \to \Omega Z^{k+1}_i$*;*
- *then the spectrum* $Q(X)$ *has* k*-th component* $(QX)_k = \operatorname{colim}_i Z^k_i$*.*

We didn't say anything about the structure maps of the spectrum QX above due to laziness. But in short terms, one can draw a diagram with upper row being the sequence $(Z^k_i)_{i\in\mathbb{N}}$ and the bottom row being $\Omega(Z^{k+1}_i)_{i\in\mathbb{N}}$. The colimit then gives the structure maps. We should at least verify that the functor Q has Ω-spectra as output.

Proposition 19. *For any spectrum* X*,* $Q(X)$ *is a* Ω*-spectrum.*

Proof. The first thing we need is to realize that part of the construction of $Q(X)$ is cofibrant replacing X. If one analyzes the inductive construction of $Q(X)$, it is possible to see how it is exactly taking a colimit while doing a CW-approximation of X as in Theorem 15.

So we can assume X is a CW-spectrum and then $Q(X)_k = \operatorname{colim}_i \Omega^i X_{k+i}$ and since S^1 is compact, $\Omega(Q(X)) = \operatorname{Map}(S^1, Q(X)) = \operatorname{Map}(S^1, \operatorname{colim}_i \Omega^i X_{k+i}) \cong \operatorname{colim}_i \operatorname{Map}(S^1, \Omega^i X_{k+i}) = \operatorname{colim}_i \Omega^i\Omega(X_{k+i}) = Q(\Omega(X))$. □

There is also a natural transformation $\eta : \mathrm{id}_{\mathbf{Sp}} \Rightarrow Q$ where $\eta_X : X \to QX$ in the level k is the map given by the universal property of the colimit. Remember that $X_k = Z^k_0$ and $(QX)_k$ is the colimit of $Z^k_0 \to Z^k_1 \to \cdots$. The claim is that $\eta_X : X \to QX$ induces isomorphisms on homotopy groups of spectra.

Lemma 20. *The morphism* $\eta_X : X \to QX$ *induces isomorphisms* $\pi_k(X) \to \pi_k(QX)$ *in the (stable) homotopy groups of spectra.*

Proof. The induced map η^*_X in the k-th homotopy groups is the colimit of π_k applied to the sequences $Z^0_n \to \Omega Z^1_{n-1} \to \cdot \to \Omega^n Z^n_0 = \Omega^n X_n$ where each map is the weak equivalence in the factorization of $Z^j_i \to \Omega Z^{j+1}_i$ provided by the model structure, as discussed in Definition 18 (the definition of the spectrification functor).

Then, the loop space-suspension adjunction provides

$$\operatorname{colim}_i \pi_{k+i}(X_i) \cong \operatorname{colim}_i \pi_k(\Omega^i X_i)$$

while the mentioned chain of weak equivalences provides the isomorphisms in π_k between $\pi_k(\Omega^i X_i)$ and $\pi_k(Z^0_i)$. At the same time, the compactness of S^k ensures that

$$\operatorname{colim}_i \pi_k(Z^0_i) \cong \pi_k(\operatorname{colim}_i Z^0_i) = \pi_k((QX)_0)$$

.

□

There are some details we are skipping because they fall heavily on the technical side. If we gather everything we did and blackbox some lemmas, we arrive at the following theorem.

Theorem 21. *The pair (Q, η) given by the spectrification functor and the natural transformation $\eta : \mathrm{id} \Rightarrow Q$ satisfies the conditions of Theorem 17.*

The model structure arising from this theorem is called the ***stable model structure*** *of spectra and it has the following properties*

- *weak equivalences are maps inducing isomorphisms on homotopy groups of spectra (these are called* ***stable weak equivalences****);*
- *fibrant objects are Ω-spectra;*
- *cofibrant objects are among the ones for the strict model structure;*
- *in particular, fibrant-cofibrant objects are CW spectra that are also Ω-spectra.*

Whitehead's theorem implies, in particular, that stable and strict weak equivalences agree for CW-spectra that are also Ω-spectra. In reality, something stronger holds, one only needs the spectra in question to be Ω-spectra.

The situation here mirrors in many ways the relation between the Quillen and the Ström model structures for topological spaces. We have some preliminary notion of weak equivalence that is too strong and the more modest one coming from isomorphisms in the homotopy groups. Also, in the same way we have reasons to prefer the Quillen model structure over Ström's one, we favour the stable model structure over the strict one.

Some reasons for that were already pointed. For example, the fact that Ω-spectra are more interesting than ordinary ones (e.g., they appear in Brown's representability and they are the *spectrum objects* in the category of spectra). But there is another reason we will briefly discuss now. The adjunction in Proposition 12 is upgraded to a Quillen equivalence when we change the model structure from the strict to the stable one.

Why would that be a big deal? Because this is the required condition for a model category to become a **stable model category**. A stable model category is one where we have things like suspensions and loop spaces and they define equivalences in the homotopy category. The precise notions of suspension and loop space are as follows.

Definition 22 (Suspension object)**.** *Given a model category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the* ***suspension object of*** *X is the homotopy pushout*

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array} .$$

Definition 23 (Loop space object)**.** *Given a model category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the* ***loop space object of*** *X is the homotopy pullback*

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} .$$

These constructions of suspension objects and loop space objects define endofunctors in the relevant model category.

Definition 24 (Stable model category)**.** *A pointed model category $\mathcal{C}$ is called* ***stable*** *if the derived suspension and loop space object functors*

$$\Sigma : \mathrm{Ho}(\mathbf{Sp}) \rightleftarrows \mathrm{Ho}(\mathbf{Sp}) : \Omega$$

are inverses of each other.

The story goes that the functors Σ and Ω realize $\mathbf{Sp}$ as a stable model category.

Theorem 25. *There is a Quillen equivalence*

$$\Sigma : \mathbf{Sp} \rightleftarrows \mathbf{Sp} : \Omega$$

when $\mathbf{Sp}$ *is equipped with the stable model structure.*

The point is that the suspension and loop space objects coincide with our old suspension and loop space functors in spectra. The fact that the stable model structure is a stable model category is very useful. For instance, the homotopy category receives a triangulated category structure. That allows for one to talk about exact sequences of spectra in a very natural way and, hence, of exact functors.

One more important fact is that stable model categories are, as one would expect, the model categories incarnating stable $(\infty, 1)$-categories. In particular, the stable model category of spectra is a way of presenting the stable ∞-category of spectra. In reality, the notation $\mathbf{Sp}$ usually is used for this ∞-category. There is a crucial factor here pushing us to work with the ∞-categorical language, which is the naturality and simplicity in putting some extra structure on the involved categories. For example, trying to come up with a definition of G-equivariant spectrum is a complicated work. Indeed, there is a bunch of (non-equivalent) definitions. Although people usually stick with genuine G-spectra when doing things equivariantly, it is, in many cases possible to obtain the same results using only the weaker notion of spectrum with a G action, which is merely a functor between the infinity categories $\mathbf{B}G$ and $\mathbf{Sp}$.

References

[BF78] A. K. Bousfield and E. M. Friedlander. “Homotopy theory of -spaces, spectra, and bisimplicial sets”. In: *Geometric Applications of Homotopy Theory II*. Ed. by M. G. Barratt and M. E. Mahowald. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1978, pp. 80–130. ISBN: 978-3-540-35808-4. DOI: 10.1007/BFb0068711.

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2020. ISBN: 978-1-108-48278-3. DOI: 10.1017/9781108636575.

[Hir] P. S. Hirschhorn. *Model Categories and Their Localizations*. DOI: https://doi.org/10.1090/surv/099.

[Hov] M. Hovey. *Model Categories*. DOI: http://dx.doi.org/10.1090/surv/063.

[Man+01] M. A. Mandell et al. “Model Categories of Diagram Spectra”. In: *Proceedings of the London Mathematical Society* 82.2 (Mar. 2001), pp. 441–512. ISSN: 00246115. DOI: 10.1112/S0024611501012692. URL: http://doi.wiley.com/10.1112/S0024611501012692.

[May98] J. P. May. “Brave new worlds in stable homotopy theory”. In: *Homotopy theory via algebraic geometry and group representations*. Evanston, United States: American Mathematical Society, 1998, pp. 193–212. ISBN: 0-8218-0805-2.