A quick note on cohomology of (quasi)coherent sheaves and Serre's criterion for affiness

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December 2023

Introduction

Sheaf cohomology is a powerful tool in algebraic geometry. It can be used in many different ways, like for obstruction theory and results like Serre's duality and all the versions of Riemann-Roch.

In this paper we focus on a particular use of sheaf cohomology, which is in classifying affine schemes. The goal of the text is to prove Serre's criterion for affiness, a result giving necessary and sufficient conditions for a Noetherian scheme to be affine, based on the cohomology of coherent and quasi-coherent sheaves. The statement of the theorem is the following.

Theorem. *For X a Noetherian Scheme, the following are equivalent:*

1. *X is affine;*
2. $\mathrm{H}^n(X, \mathcal{F}) = 0$ *for all $n > 0$ and quasi-coherent sheaf $\mathcal{F}$;*
3. $\mathrm{H}^1(X, \mathcal{I})$ *for all coherent sheaves of ideals $\mathcal{I}$.*

The goal of this document is to present a basic use of sheaf cohomology, in the form of a summary of the proof of an affiness criterion. We are going to cover the basics of coherent and quasi-coherent sheaves and some results concerning their cohomology.

The first section is dedicated to the presentation of the notions of coherent and quasi-coherent sheaves. We base our discussion on chapters 2 and 3 of [Har77] and the interested reader can check there for more details. We define the sheaf associated to a module and we prove some basic results. Then we present the concepts of coherent and quasi-coherent sheaves together with an important theorem on the matter.

The proof of the main theorem is covered in section two. We prove some necessary lemmas to obtain the criterion for affiness.

Conventions

For us, a ring is a commutative ring with unity and morphisms of rings map units to units. By an R-module we mean an abelian group with an action of R, where the unit of R acts trivially.

We don't particularly care about the sides of our modules. That means this text was not written with the care of making the notation consistent with the modules being left or right modules. In practice, if M is an R-module, we ask the reader the patience to accept both $r \cdot m$ and $m \cdot r$, independently of M being a right, left or bimodule. If it makes the reader comfortable, one can choose to make all modules in this text right modules and every time some inconsistent notation appears, swap the module and the ring elements to make it right. This should not cause any issues as there are not substantial calculations involving the elements of some module.

1 Quasi-coherent and coherent sheaves

In this section we are going to introduce the concepts of coherent and quasi-coherent sheaves. The take-off of the idea of (quasi)coherent sheaves is that they are part of an algebraic analogous of Serre-Swan Theorem on vector bundles. The category of quasi-coherent modules over some affine scheme $(X, \mathcal{O}_X)$ is equivalent to the category of $\Gamma(X, \mathcal{O}_X)$-modules. Similarly to what happens with vector bundles (which come with an equipped sheaf of sections), global sections are responsible for this equivalence.

Maybe a slogan to have in mind is that vector bundles are not suitable for homological algebra, as they do not form an abelian category. Quasi-coherent sheaves can be though of as a generalization of vector bundles, one that does form an abelian category.

First of all, we need to define the sheaf associated to a modules, as the actual definition of (quasi)coherent sheaves is basically a sheaf that locally is a sheaf associated to a module.

Definition 1 (Sheaf associated to M)**.** *Let A be a ring and M be an A-module. We define a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ by specifying what it does on a basis of $\operatorname{Spec} A$. For each $f \in \operatorname{Spec} A$, set $\widetilde{M}(D(f)) = M_f$. The sheaf $\widetilde{M}$ is called the* ***sheaf associated to*** *M on $\operatorname{Spec} A$.*

Now, we can show some basic results about it.

Proposition 2. *Given an A-module M and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the stalk of $\widetilde{M}$ in $\mathfrak{p}$ is the localization*

$$(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}.$$

Proof. Interpreting the stalk of $\widetilde{M}$ at $\mathfrak{p}$ as compatible germs, we define a map

$$\begin{aligned}\operatorname{colim}_{f\notin\mathfrak{p}} \widetilde{M}(D(f)) = (\widetilde{M})_{\mathfrak{p}} &\to M_{\mathfrak{p}}\\ (s,U) &\mapsto s(\mathfrak{p}).\end{aligned}$$

Showing surjectivity is the easier part, as any element in $M_{\mathfrak{p}}$ can be written as a fraction m/f with $f \notin \mathfrak{p}$, by definition. We can then consider the germ $(m/f, D(f))$ as $\mathfrak{p} \in D(f)$, which defines a section being mapped to $m/f \in M_{\mathfrak{p}}$.

Now for injectivity, consider two compatible germs at $\mathfrak{p}$, denote them by a/f and b/g with $f, g \notin \mathfrak{p}$. From the definition of localization, there is some $h \notin \mathfrak{p}$ such that $h \cdot (ag - bf) = 0$. In particular, as $h \neq 0$, in any localization $M_{\mathfrak{q}}$ where h is not a divisor of zero, $a/f = b/g$ holds. That happens, for instance, if h is a unit, a case that is comprehended in localizations at ideals in $D(h)$. Hence, the equation $h \cdot (ag - bf) = 0$ implies $a/f = b/g$ in localizations of M at any ideal where $1/f, 1/g$ are defined and h is a unit, which is precisely $D(f) \cap D(g) \cap D(h)$. But this set is an open neighborhood of $\mathfrak{p}$ as none of f, g, h are in $\mathfrak{p}$ and, consequently, $a/f = b/g$ holds in some neighborhood of $\mathfrak{p}$, so the germs corresponding to a/f and b/g are identified in the stalk $(\widetilde{M})_{\mathfrak{p}}$. □

Proposition 3. *Let M be an A-module and $\widetilde{M}$ the associated sheaf. The global sections of $\widetilde{M}$ are $\Gamma(\operatorname{Spec} A, \widetilde{M}) = M$.*

Proof. Setting $f = 1$, we have $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : 1 \notin \mathfrak{p}\} = \operatorname{Spec} A$, so $\widetilde{M}(D(f)) = \widetilde{M}(\operatorname{Spec} A) = M_1 = M$. □

The two theorems above are somehow expected from the way we defined the sheaf associated to M. The definition is pretty much the same as the structure sheaf on $\operatorname{Spec} A$, replacing A with M. Indeed, the proofs are the same as the ones used for the structure sheaf.

Now, we finally define coherent and quasi-coherent sheaves. As previously stated, they are to be seen as sheaves that locally look like sheaves associated to some modules.

Definition 4 (Quasi-coherent and coherent sheaf)**.** *If $(X, \mathcal{O}_X)$ is a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is a **quasi-coherent sheaf** if there is an affine open cover $\{U_i\}_i$ with $U_i = \operatorname{Spec} A_i$ and $\mathcal{F}|_{U_i}$ is isomorphic to a sheaf associated to an A_i-module.*

*If each of these A_i-modules is finitely generated, then $\mathcal{F}$ is a **coherent sheaf**.*

With the concepts of coherent and quasi-coherent in hand, we can see how categories of quasi-coherent sheaves are related to categories of modules, via the following theorem, that we state with no proof.

Theorem 5. *Given an affine scheme $X = \operatorname{Spec} A$, the global sections functor $\Gamma(X, -)$ and the functor $\widetilde{(-)}$ giving the associated sheaf to a module constitutes an equivalence between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules.*

2 Cohomology of affine schemes

In this section, we prove our main theorem. We just need to go through some necessary lemmas. The first interesting result we would like to use is the one establishing a bijection between maps into an affine scheme and maps from the corresponding ring. We first show this theorem in the case the maps into the affine scheme are also coming from an affine scheme.

Lemma 6. *There is a bijection*

$$\beta : \mathrm{Hom}_{\mathbf{Ring}}(A, B) \to \mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A)$$

between morphisms of affine schemes and morphisms of the corresponding rings.

Proof. Given a morphism of rings $\phi : A \to B$, we define the maps $(f, f^{\#})$ in the following manner:

- let f be the preimage function ϕ^{-1}, so $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$;
- testing the continuity of f can be done on the basic open sets and then we have $f^{-1}(D(g)) = D(\phi(g))$, which is also open;
- we define $f^{\#}$ in the distinguished opens by setting it to be the map

  $$f^{\#}(D(g)) : \mathcal{O}_{\operatorname{Spec} A} = A_g \to B_{\phi(g)} = \mathcal{O}_{\operatorname{Spec} B}(D(\phi(g))) \cong f_* \mathcal{O}_{\operatorname{Spec} B}(D(g))$$

  obtained by localization;
- set $\beta(\phi)$ to be $(f, f^{\#})$ as above.

One can see the injectivity of β as soon as one sees that for two different morphisms of rings, the corresponding inverse image functions are distinct. For surjectivity, take any morphism $(f, f^{\#})$ between the schemes $\operatorname{Spec} B$ and $\operatorname{Spec} A$. Applying the global sections functor gives a morphism $\phi : A \to B$ (just take the $f^{\#}(\operatorname{Spec} A) : \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A \to B = \Gamma(f^{-1}(\operatorname{Spec} A), \mathcal{O}_{\operatorname{Spec} B})$).

The claim is that testing $\phi = f^{\#}(\operatorname{Spec} A)$ against β gives back the morphism of schemes $(f, f^{\#})$ we started with. This is the observation that as ϕ is on global sections, it must be compatible with restrictions. In particular, the restriction of ϕ to distinguished opens must coincide with the morphisms obtained by evaluating $f^{\#}$ at the corresponding distinguished opens. But the restriction of ϕ to distinguished opens is precisely the localization of ϕ, which is precisely the definition of the map of sheaves $\mathcal{O}_{\operatorname{Spec} A} \to f_* \operatorname{Spec} \mathcal{B}$ we get when applying β to ϕ. ☐

Now we upgrade the previous result replacing $\operatorname{Spec} B$ by a general scheme X. As usual, we use the lemma above in each open of the affine cover of X and glue everything.

Proposition 7. *Let $(X, \mathcal{O}_X)$ be a scheme. Then, for any ring A, there is a bijection*

$$\alpha : \mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}\, A) \to \mathrm{Hom}_{\mathbf{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Proof. Lemma 6 is the special case for X affine. We will cover X with an affine cover and use the affine case to obtain the general one. Define α in the same way as one does for the affine case (see the proof Lemma 6 for a reminder) and take an affine cover $\{U_i = \mathrm{Spec}\, B_i\}$ of X, for which we know the maps

$$\alpha_i : \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, B_i, \mathrm{Spec}\, A) \to \mathrm{Hom}_{\mathbf{Ring}}(A, B_i)$$

are isomorphisms, being α_i obtained by restricting α to each affine cover, so they take the role of α for the affine cases, as in Lemma 6. This way, we get commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}\, A) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbf{Ring}}(A, \Gamma(X, \mathcal{O}_X)) \\ \big\downarrow{\scriptstyle |_{U_i}} & & \big\downarrow{\scriptstyle \mathrm{res}_{X,U_i}} \\ \mathrm{Hom}_{\mathbf{Sch}}(U_i, \mathrm{Spec}\, A) & \xrightarrow{\alpha_i} & \mathrm{Hom}_{\mathbf{Ring}}(A, B_i) \end{array}$$

one for each i. With this, we can show α is a bijection as well. For injectivity, consider two morphisms $f = (f, f^{\#})$ and $g = (g, g^{\#})$ from X to $\mathrm{Spec}\, A$ such that $\alpha(f) = \alpha(g)$. Then, restricting f, g and $\alpha(f)$ and $\alpha(g)$, by the commutativity of the diagrams above, we obtain for each i, morphisms of rings

$$\alpha_i(f|_{U_i}) = \mathrm{res}_{X,U_i}(\alpha(f)) = \mathrm{res}_{X,U_i}(\alpha(g)) = \alpha_i(g|_{U_i}).$$

But each α_i is injective, so $f|_{U_i} = g|_{U_i}$ for all i. By identity (of morphisms of schemes), since the U_i form a cover for X, $f = g$.

Now, for surjectivity, we start with a map of rings $\phi : A \to \Gamma(X, \mathcal{O}_X)$ and restriction gives for each i a morphism of rings $\phi_i : A \to B_i$. Using the surjectivity of the α_i, we can find morphisms of schemes $f_i : U_i \to \mathrm{Spec}\, A$ such that $\alpha_i(f_i) = \phi_i$. Furthermore, since each U_i is affine and the proposition we are proving holds for the affine case, the ϕ_i's are forced to agree on intersections of the U_i's. In more precise terms,

$$\phi_{ij} := \mathrm{res}_{U_i, U_i \cap U_j}(\phi_i) = \mathrm{res}_{U_j, U_i \cap U_j}(\phi_j),$$

and as $U_i \cap U_j$ is affine, we can get a bijection α_{ij}, lifting ϕ_{ij} to some f_{ij} that can be obtained by restricting f_i and f_j. So we can use gluability to construct a map $f : X \to \mathrm{Spec}\, A$, giving the surjectivity of α.

□

An interesting remark about Proposition 7 is that it can be restated in terms of an adjunction between Spec and the dual of global sections $\Gamma()$, defined in the dual category of rings $\mathbf{Ring}^*$.

Definition 8 (Quasi-compact scheme)**.** *A scheme $(X, \mathcal{O}_X)$ is said to be* ***quasi-compact*** *if X is compact as a topological space.*

We are now approaching our main result. We want to prove a more technical lemma that gives a criterion for affiness, one that will be used during the proof of Serre's criterion. We first fix the notation below and start our laborious work.

Notation: For a global section $f \in \Gamma(X, \mathcal{O}_x)$, we denote by X_f the set $\{x \in X : f_x \notin \mathfrak{m}_x\}$.

Lemma 9. *Let $(X, \mathcal{O}_X)$ be a scheme and $A = \Gamma(X, \mathcal{O}_X)$ its ring of global sections. If X has a finite affine cover $\{U_i\}$ with each intersection $U_i \cap U_j$ being quasi-compact, then for any $f \in A$, we have $A_f \cong \Gamma(X_f, \mathcal{O}_{X_f})$.*

Proof. Define the map

$$\begin{aligned} \phi :& A_f \to \Gamma(X_f, \mathcal{O}_{X_f}) \\ & a/f^n \mapsto \operatorname{res}_{X,X_f}(a)/\operatorname{res}_{X,X_f}(f)^n. \end{aligned}$$

We will show that ϕ, as above, is a bijection.

For the injectivity part, suppose $a/f^n \in A$ is such that $\operatorname{res}_{X,X_f}(a/f^n) = 0$. Write X as a finite union of affine schemes using its quasi-compactness $X = U_1 \cup \cdots \cup U_k$ and consider the elements $\operatorname{res}_{X,U_i \cap X_f}(a/f^n)$ for different i's.

Each $U_i \cong \operatorname{Spec} B_i$ is affine, so $f_i = \operatorname{res}_{X,U_i}(f)$ is a global section of an affine scheme. Then, unpacking the established notation, $(U_i)_{f_i} = \{\mathfrak{p} \in \operatorname{Spec} B_i : f_i \notin m_{\mathfrak{p}}\}$ is the same as the set of ideals $\mathfrak{p}$ in U_i where $f_i \notin (B_i)_{\mathfrak{p}}$, that is, the distinguished open $D(f_i)$ for $\operatorname{Spec} B_i$. As $\operatorname{res}_{X,X_f}(a/f^n) = 0$, composition with further restrictions still gives 0, so $\operatorname{res}_{X,X_f \cap U_i}(a/f^n) = 0$ as well. The detail here is that $\operatorname{res}_{X,X_f \cap U_i}(a/f^n)$ lives in $\Gamma(U_i, \mathcal{O}_X) \cong (B_i)_{f_i}$, so being 0, by the definition of localization at f_i means that $f_i^l a = 0$ for some l in B_i.

Of course this process can be done for each i so we get many different exponents for which the terms $f_i^l a$ will all be 0. As there are finitely many U_i's, we can choose the largest exponent m and as the U_i cover X, we can glue the f_i^m to form f^m, so $f^m a = 0$, so $a/f^n = 0$.

Now surjectivity. Take an element $a \in \Gamma(X_f, \mathcal{O}_{X_f})$, by restricting it to the intersection with opens of an affine covering $U_1 \cup \cdots U_k$, as before, we get $\operatorname{res}_{X_f, X_f \cap U_i}(a)$ living in $\Gamma(X_f \cap U_i, \mathcal{O}_{X_f}) \cong (B_i)_{f_i}$. An element of this ring is generally given by a fraction $a_i/f_i^{l_i}$. Define the sections

$$s_i = f^{(\sum_j l_j) - l_i} b_i$$

and notice that $\operatorname{res}_{U_i, X_f \cap U_i}(s_i) = \operatorname{res}_{X_f, X_f \cap U_i}(f^{\sum_j l_j} a)$, so s_i and s_j coincide in their intersections $U_i \cap U_j$. In the injectivity proof, we have seen that in this case, as U_i and U_j are affine (and then so is $U_i \cap U_j$), there is some m such that $f^m(s_i - s_j)$ in $U_i \cap U_j$. As there are finitely many U_i's and U_j's, each pair guaranteeing some m, we can choose a very large value M working for all the pairs and some extra room, yielding, as in the proof for injectivity, sections $f^M b_i \in \Gamma(U_i, \mathcal{O}_X)$.

The last step is to use gluability to form a global section $f^M b$. Then, one can divide by a sufficiently large power of f in A_f to get the element that restricts to a.

□

Lemma 10. *A scheme $(X, \mathcal{O}_X)$ is affine if, and only if, there is a finite set of global sections $\{f_1, \dots, f_n\} \subset \Gamma(X, \mathcal{O}_X)$ such that each X_{f_i} is affine and the sections $f_1, \dots, f_n$ generate $\Gamma(X, \mathcal{O}_X)$ as an ideal.*

Proof. If $(X, \mathcal{O}_X)$ is an affine scheme $X = \operatorname{Spec} A$, then $1 \in A$ is a global section and $X_1 = D(1) = X$, so this direction is done.

The other direction is not so easy. Let $\{f_1, \dots, f_n\}$ be global sections of the scheme $(X, \mathcal{O}_X)$ with $X_{f_i} \cong \operatorname{Spec} A_i$ affine and the f_i's generating $A = \Gamma(X, \mathcal{O}_X)$.

By Proposition 7, we have a map of schemes $\psi : X \to \operatorname{Spec} A$ associated with the identity $A \to \Gamma(X, \mathcal{O}_X)$.

Restricting ψ to X_{f_i} yields maps $\psi_i : X_{f_i} \to \operatorname{Spec} \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$. By hypothesis, the X_{f_i} are all affine, so Lemma 9 establishes the isomorphism between X_{f_i} and $\operatorname{Spec} \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \cong \operatorname{Spec} A_{f_i}$.

Then, X is covered by the open affines X_{f_i} due to the f_i's generating A. And the maps $X_{f_i} \to A_{f_i}$ are all compatible as they come from the restriction of ψ. Hence, we can glue all of them to get ψ back, which is then forced to be an isomorphism.

□

This finishes the more technical part. Now we do a small calculation that will be used in Serre's criterion.

Proposition 11. *If $\mathcal{F}$ is a quasi-coherent sheaf on an affine scheme $X = \operatorname{Spec} A$ where A is Noetherian, then $\mathrm{H}^n(X, \mathcal{F}) = 0$ for all $n > 0$.*

Proof. Given a quasi-coherent sheaf $\mathcal{F}$ on $X = \operatorname{Spec} A$, we let $M = \Gamma(X, \mathcal{F})$ be its A-module of global sections and then we choose an injective resolution

$$0 \to M \to I^\bullet$$

for M. From the equivalence of categories described in Theorem 5, we may replace $\mathcal{F}$ by $\widetilde{M}$. Now, considering the sheaves associated to the modules of the resolution $I^\bullet$ we obtain a resolution for $\widetilde{M}$

$$0 \to \widetilde{M} \to \widetilde{I}^\bullet$$

that can be used to compute cohomology, as the sheaves $\widetilde{I}^i$ are flasque and flasque sheaves are acyclic. Hence, $H^0(X, \widetilde{M}) = \Gamma(X, \widetilde{M}) = M$ and $H^n(X, \widetilde{M}) = 0$ for $n > 0$.

□

That is it. We are ready to prove our main theorem. Serre's criterion for affineness is a result concerning Noetherian schemes. The definition of a Noetherian scheme is given below, just before the statement and proof of the theorem.

Definition 12 (Noetherian Scheme)**.** *A scheme $(X, \mathcal{O}_X)$ is **locally Noetherian** if every point $x \in X$ has an affine neighborhood $U \cong \operatorname{Spec} R$ with R a Noetherian ring. If the scheme X is also quasi-compact, then it is a **Noetherian scheme**.*

Theorem 13. *If X is a Noetherian Scheme. The following statements are equivalent.*

1. *X is an affine scheme;*
2. *$\mathrm{H}^n(X, \mathcal{F}) = 0$ for all $n > 0$ and quasi-coherent sheaf $\mathcal{F}$;*
3. *$\mathrm{H}^1(X, \mathcal{I})$ for all coherent sheaves of ideals $\mathcal{I}$.*

Proof. The implication (1) $\Rightarrow$ (2) is Proposition 11. Implication (2) $\Rightarrow$ (3) follows from the definition, as all coherent sheaves are quasi-coherent.

We only need to show (3) $\Rightarrow$ (1). For that, consider a closed point $P \in X$ and an open affine neighborhood U of P. Define Y to be $X \setminus U$, so we have an exact sequence of ideal sheaves

$$0 \longrightarrow \mathcal{I}_{Y \cup \{P\}} \longrightarrow \mathcal{I}_Y \longrightarrow k(P) \longrightarrow 0$$

with quotient being a skyscraper sheaf at P with values in $(O_X)_P/\mathfrak{m}_P$. This way, we obtain an exact sequence on cohomology

$$\Gamma(X, \mathcal{I}_{Y \cup \{P\}}) \to \Gamma(X, \mathcal{I}_Y) \to \Gamma(X, k(P)) \to H^1(X, \mathcal{I}_{Y \cup \{P\}}) \to \cdots .$$

But we are working under the hypothesis that the first cohomology of coherent sheaves of ideals vanishes, so the last term is simply 0, meaning the map $\Gamma(X, \mathcal{I}_Y) \to \Gamma(X, k(P))$ is surjective. Consequently, there is some element $f \in \Gamma(X, \mathcal{I}_Y)$ which is mapped to $1 \in k(P)$, so $f_P \notin \mathfrak{m}_P$ and, thus, $X_f \subset U$. More precisely, if we let $g = \operatorname{res}_{X,U}(f)$, then $X_f = U_g$, which is affine because U is. In synthesis, every closed point admits an affine open of the form X_f.

Now, what we need to do is take all the closed points of X and select, using the quasi-compactness of X, a finite amount, yielding a finite set $f_1, f_2, \ldots, f_n$ of global sections of an affine cover $X_{f_1}, X_{f_2}, \ldots, X_{f_n}$. This way, we can define a morphism of sheaves $\alpha : (\mathcal{O}_X)^n \to \mathcal{O}_X$ that maps $(a_1, \ldots, a_n) \in \Gamma(U, \mathcal{O}_X)^n$ to $\sum a_i \operatorname{res}_{X,U_i}(f_i)$. The fact that the sets X_{f_i} cover X guarantees its surjectivity. We can fit this map into an exact sequence with its kernel:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^n \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow 0.$$

Looking at the long exact sequence on cohomology, at the moment we observe that $H^1(X,\mathcal{F}) = 0$, we can conclude that the map $\Gamma(X,\alpha)$ is a surjection and we conclude that the $f_1,\dots,f_n$ generate $\Gamma(X,\mathcal{O}_X)$, from where we can apply Lemma 10.

To see why $H^1(X,\mathcal{F})$ is trivial, one can look at the sequence of inclusions

$$\mathcal{F}\cap\mathcal{O}_X \subset \mathcal{F}\cap\mathcal{O}_X^2 \subset \cdots \subset \mathcal{F}\cap\mathcal{O}_X^n = \mathcal{F}$$

giving rise to short exact sequences

$$0 \longrightarrow \mathcal{F}\cap\mathcal{O}_X^k \longrightarrow \mathcal{F}\cap\mathcal{O}_X^{k+1} \longrightarrow \frac{\mathcal{F}\cap\mathcal{O}_X^{k+1}}{\mathcal{F}\cap\mathcal{O}_X^k} \longrightarrow 0,$$

one for each k. Thus, for each k there is a long exact sequence in cohomology. The interesting thing is the exactness of the part

$$H^1(X,\mathcal{F}\cap\mathcal{O}_X^k) \longrightarrow H^1(X,\mathcal{F}\cap\mathcal{O}_X^{k+1}) \longrightarrow H^1(X,\frac{\mathcal{F}\cap\mathcal{O}_X^{k+1}}{\mathcal{F}\cap\mathcal{O}_X^k}).$$

If we could show that $H^1(X,\mathcal{F}\cap\mathcal{O}_X^k) = 0$, then, since $\mathcal{F}\cap\mathcal{O}_X^{k+1}/\mathcal{F}\cap\mathcal{O}_X^k$ is a coherent sheaf of ideals, the left and right terms in the exact sequence are trivial by hypothesis and we conclude that $H^1(X,\mathcal{F}\cap\mathcal{O}_X^{k+1}) = 0$ as well. Together with the fact that $\mathcal{F}\cap\mathcal{O}_X$ is a coherent sheaf of ideals, we have the base case for our induction.

This way, as $\mathcal{F}\cap\mathcal{O}_X^n = \mathcal{F}$, we get that $H^1(X,\mathcal{F})$ is trivial, as desired. □

We are done. It is interesting to note that there are other versions of this theorem, with loosened requirements. One can prove the same result by requiring only quasi-compactness instead of full Noetherianess. There are many versions with different conditions both in the scheme and in the coherent and quasi-coherent sheaves, but the proofs may differ.

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