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Dimensão de Kimura de motivos de Chow

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Resumo

MACIEL, S. **Dimensão de Kimura de motivos de Chow**. 2024. 44 p. Monografia (Bacharelado em Matemática) – Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2º Semestre de 2024.

Nesse trabalho, apresentamos a construção da categoria dos motivos de Chow, as conjecturas padrão e o conceito de dimensão de Kimura, introduzida, de forma independente, por Kimura e O'Sullivan. Em seguida, discutimos algumas propriedades de motivos de dimensão finita, como estabilidade por soma e produto tensorial, e provamos que o motivo de Chow de curvas tem dimensão finita, apresentando algumas consequências desse fato para a teoria dos motivos.

Palavras-chave: Dimensão de Kimura. Motivos de Chow.

ABSTRACT

MACIEL, S. **Kimura dimension of Chow motives**. 2024. 44 p. Monografia (Bacharelado em Matemática) – Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2º Semestre de 2024.

In the present work, we construct the category of Chow motives and show the standard conjectures. Following that, we define the notion of Kimura dimension of a motive, introduced, independently, by Kimura and O'Sullivan. Then, we present some of the main properties of finite-dimensional motives, such as stability under sum and tensor product, and prove that the motive of a curve is finite-dimensional. We end by discussing some consequences of this result.

Keywords: Kimura dimension. Chow motives.

Contents

Introduction

The idea of a motive was introduced in 1964 by Grothendieck in a letter to Serre, where he mentions the term in a rather mysterious way, as something that "looks like the ℓ-adic cohomology group of an algebraic scheme, but considered as being independent of ℓ". At the time, Grothendieck himself recognizes that he does not yet know how to appropriately define the category of motives, but that the "yoga of motives" is already taking shape. In fact, in the same letter where he introduced the term, even with no concrete representation of the idea of motives, he proceeds to make speculative comments on the utility of such a notion. For example, Grothendieck explains that he expects "good cohomology theories" (which we call Weil cohomology theories today) to factor through the category of motives. He also suggests the existence of deep relations of the category of motives with the Grothendieck ring of varieties. In fact, he predicts the existence of a map from the Grothendieck ring of varieties to the category of motives.

The concept of motive rapidly became prominent in the Grothendieck-Serre correspondence and started to be widely used in algebraic geometry. From there, the theory of motives has made appearances in many different contexts, with a wide variety of applications, and has had lasting influence on a large portion of algebraic geometry.

Grothendieck believed that the theory of motives would make it possible to prove the Weil conjectures (as he explains in [Gro]). The most evident manifestation of this ambition is the formulation of the standard conjectures, a collection of conjectures on the structure of algebraic cycles that would have strong implications for the theory of motives. So strong, in fact, that they would imply the Weil conjectures. One could say that Grothendieck was to know that Deligne's proof of the Weil conjectures did not use the theory of motives (as visible in [Gro] and registered in [Sch]). Even though the Weil conjectures have already been proven, the standard conjectures, despite partial progress, remain open, along with many other fundamental questions about motives that are still completely or only partially answered.

The purpose of this work is to introduce the reader to the notion *Kimura-O'Sullivan dimension* of a pure motive and tropes about the subject. In the first chapter, we present the classical theory of Chow motives, constructing the category of pure motives and discussing important properties of this category, as well as some fundamental features of its objects. Afterwards, we dedicate some time to state the standard conjectures and explain the relation between them.

In the second chapter, we encounter the notion of Kimura-O'Sullivan dimension of a motive. Then, we start proving many results about finite-dimensional motives. Using this set up work, we will be able to make statements about the dimension of certain motives: we will show that the motive associated to any variety dominated by products of curves is finite-dimensional.

Chapter 1

The category of pure motives

1.1 Construction and elementary properties

In this section, we will construct the category of Chow motives and prove some general facts about it. We direct the reader to [Ful98] for a complete introduction to adequate equivalence relations, Chow groups, and intersection theory. Denote by $\mathbf{SmProj}(k)$ the category of smooth projective varieties over the field k. Given an adequate equivalence relation $\sim$, e.g., rational equivalence, we write $C_\sim(X)$ for the ring of cycles on X up to the relation $\sim$. This ring has a natural grading given by $C^i_\sim(X)$, the cycles of codimension i. For example, if $\sim$ is rational equivalence, $C_\sim(X)$ is just the Chow ring $\mathrm{CH}(X)$. We also write $C^i_\sim(X;\mathbb{Q})$ for $C^i_\sim(X)\otimes\mathbb{Q}$.

Definition 1.1.1 (Correspondence)**.** For $X, Y \in \mathbf{SmProj}(k)$, define

$$\mathrm{Corr}_\sim(X,Y) := C_\sim(X\times Y;\mathbb{Q}).$$

We call an element of $\mathrm{Corr}_\sim(X,Y)$ a **correspondence** from X to Y.

From now on, we will often omit $\sim$ in both $\mathrm{Corr}_\sim$ and $C_\sim$, and work with an implicit equivalence relation. Similarly, since we will only work with rational coefficients, we will omit $\mathbb{Q}$ in $C(X;\mathbb{Q})$ and write simply $C(X)$. Quintessential instances of correspondences are graphs of morphisms between algebraic varieties, and we will think of a correspondence as a type of generalized morphism between varieties. Following this idea, we often write a correspondence $f \in \mathrm{CH}(X\times Y)$ as $f : X \longrightarrow Y$ and we *intuitively* think of $\mathrm{Corr}(X,Y)$ as a thickening of $\mathrm{Hom}_{\mathbf{SmProj}}(X,Y)$. The canonical isomorphism $C(X\times Y)\cong C(Y\times X)$ immediately allows us to view $f\in\mathrm{Corr}(X,Y)$ as a correspondence from Y to X as well, but we make the choice to differentiate between these two correspondences: we will instead write f^{T} for f viewed as a correspondence from Y to X.

We may compose correspondences as follows. If $f \in \mathrm{Corr}(X, Y)$ and $g \in \mathrm{Corr}(Y, Z)$, then $g \circ f$ is defined to be

$$g \circ f := [\pi_{X \times Z}]_* [\pi^*_{X \times Y}(f) \cdot \pi^*_{Y \times Z}(g)],$$

where $\cdot$ stands for the intersection product in $C(X \times Y \times Z)$. In some situations (especially in longer computations), to improve readability, we will write gf for the composition $g \circ f$. We give a special name to the correspondences that are idempotent with respect to composition.

Definition 1.1.2 (Projector)**.** A **projector** is a correspondence $p \in \mathrm{Corr}(X, Y)$ satisfying $p \circ p = p$.

Definition 1.1.3 (Degree of a correspondence)**.** Let $X, Y \in \mathbf{SmProj}(k)$ be connected varieties with $\dim X = m$, then we define

$$\mathrm{Corr}^d(X, Y) := C^{m+d}(X \times Y).$$

Elements of $\mathrm{Corr}^d(X, Y)$ are then said to have degree d.

If X and Y have multiple connected components $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$, then we have the equality $C(X \times Y) = \oplus_{i,j} C(X_i \times Y_j)$ between graded rings. So letting $m = \dim X$ the maximum of the dimensions of the X_i, we may say a correspondence from X to Y has degree d if it belongs to $C^{d+m}(X \times Y)$.

Let X, Y, Z be varieties of dimension m, n, r, respectively, then the following facts are immediate from the definition:

(i) if $f \in \mathrm{Corr}^d(X, Y)$, then $f^{\mathrm{T}} \in \mathrm{Corr}^{d+m-n}(Y, X)$;

(ii) if $f \in \mathrm{Corr}^d(X, Y)$ and $g \in \mathrm{Corr}^l(Y, Z)$, then $g \circ f \in \mathrm{Corr}^{d+l}(X, Z)$;

(iii) if $p \in \mathrm{Corr}(X, X)$ is a projector, then p has degree 0;

(iv) if $f \in \mathrm{Corr}^d(X, Y)$, then $\Delta_Y \circ f = f$ and $f \circ \Delta_X = f$, where $\Delta_X = \{(x, x) : x \in X\}$ and $\Delta_Y = \{(y, y) : y \in Y\}$.

A correspondence $f \in \mathrm{Corr}^d(X, Y)$ induces a homomorphism on cycles

$$\begin{aligned} f_* : C^i(X) &\longrightarrow C^{i+d}(Y) \\ Z &\longmapsto f_*(Z) = [\pi_Y]_*(f \cdot \pi_X^*(Z)). \end{aligned}$$

If the implicit adequate equivalence relation is finer than homological equivalence, then $f \in \mathrm{Corr}^d(X, Y)$ also induces a homomorphism on any Weil cohomology groups (see [Jon07] for an introduction to Weil cohomology theories):

$$\begin{aligned} f_*: H^i(X) &\longrightarrow H^{i+2d}(Y) \\ \alpha &\longmapsto f_*(\alpha) = [\pi_Y]_*(\gamma(f) \cup \pi_X^*(\alpha)). \end{aligned}$$

Definition 1.1.4 (The category of motives)**.** Denote by $\mathbf{Mot}_\sim(k)$ the category whose objects are triples (X, p, m), where X is a smooth projective variety, $p \in \mathrm{Corr}_\sim(X \times X)$ is a projector, and m a natural number. Morphisms in $\mathbf{Mot}_\sim(k)$ are given by

$$\mathrm{Hom}_{\mathbf{Mot}_\sim(k)}((X, p, m), (Y, q, n)) := q \circ \mathrm{Corr}_\sim^{n-m}(X, Y) \circ p,$$

so a morphism from (X, p, m) to (Y, q, n) is a correspondences from X to Y of the form $q \circ f \circ p$, where $f \in \mathrm{Corr}_\sim^{n-m}(X, Y)$. As for Chow rings, we will omit $\sim$ in $\mathbf{Mot}_\sim(k)$.

A motive of the form $(X, p, 0)$ is called an **effective motive**. There is a functor

$$\begin{aligned} h: \mathbf{SmProj}(k)^{\mathrm{op}} &\longrightarrow \mathbf{Mot}(k) \\ X &\longmapsto (X, \Delta_X, 0) \end{aligned}$$

whose image lies in the subcategory of effective motives $\mathbf{Mot}^{\mathrm{eff}}(k)$, that is, the full subcategory whose objects are the effective motives. A morphism of smooth projective varieties $f : X \longrightarrow Y$ is mapped to the morphism of motives $\Gamma_f^{\mathrm{T}} : h(Y) \longrightarrow h(X)$, that is, the transpose of the correspondence given by the graph of f.

Another perspective on the construction of the category of motives is the following. A Karoubian (or pseudo-abelian) category is a pre-additive category where every idempotent has kernel and cokernel. The Karoubian completion of a category is a procedure to construct a Karoubian category from any pre-additive category. One can form a category $\mathrm{Corr}(k)$ whose objects are smooth projective varieties and morphisms are degree zero correspondences. Then, the category of effective motives is defined to be the Karoubian completion of $\mathrm{Corr}(k)$. In fact, a morphism of effective motives $f : (X, p, 0) \longrightarrow (Y, q, 0)$ is a correspondence of the form $f = qf'p$, with $f' \in \mathrm{Corr}^0(X, Y)$, which is equivalent to $q \circ f = f = f \circ p$. This is exactly the usual description of the construction of the Karoubian completion of a category.

To arrive at the category of motives from the one of effective motives constructed as the Karoubian completion of $\mathrm{Corr}(k)$, one only needs to observe that there is a tensor product in $\mathbf{Mot}^{\mathrm{eff}}(k)$ and formally adding an inverse for the motive $(\mathbb{P}^1, \mathbb{P}^1 \times \{*\}, 0)$ with respect to this tensor product gives $\mathbf{Mot}(k)$. This story will become clear after we describe the monoidal structure on $\mathbf{Mot}(k)$.

The category $\mathbf{Mot}(k)$ is an enlargement of $\mathbf{Mot}^{\text{eff}}(k)$ that accounts for twists of effective motives, which are meant to incorporate Tate twists of Hodge structures. Moreover, $\mathbf{Mot}(k)$ is a $\mathbb{Q}$-linear category: given two motives $M = (X, p, m)$ and $N = (Y, q, n)$, $\text{Corr}^{n-m}(X, Y) = C^n(X \times Y)$ is a $\mathbb{Q}$-vector space (remember we work with rational coefficients). Given two maps of motives $f = qf'p$ and $g = qg'p$, linearity of composition of correspondences implies that

$$qf'p + qg'p = q(f' + g')p,$$

so $q \circ \text{Corr}^{n-m}(X, Y) \circ p$ is a subspace of $\mathbb{Q}$-vector, giving $\text{Hom}_{\mathbf{Mot}(k)}(M, N)$ the structure of a $\mathbb{Q}$-vector space.

Definition 1.1.5 (Tensor product of motives)**.** Given two motives $M = (X, p, m)$ and $N = (Y, q, n)$, their **tensor product** is defined to be

$$M \otimes N = (X \times Y, p \times q, m + n).$$

This tensor product turns $\mathbf{Mot}(k)$ into a symmetric monoidal category. The unit is the motive of a point $\mathbf{1} = h(\text{Spec}\, k) = (\text{Spec}\, k, \Delta_{\text{Spec}\, k}, 0)$. Fixing a point $e \in \mathbb{P}^1$, we define the **Lefschetz motive** to be

$$\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times e, 0),$$

and the **Tate motive** to be

$$\mathbb{T} = (\text{Spec}\, k, \Delta_{\text{Spec}\, k}, 1).$$

Then, the morphism of motives $\Delta_{\text{Spec}\, k} \circ (\mathbb{P}^1 \times \text{Spec}\, k) \circ (\mathbb{P}^1 \times e) : \mathbb{L} \longrightarrow (\text{Spec}\, k, \Delta_{\text{Spec}\, k}, -1)$ is an isomorphism: its inverse is $(\mathbb{P}^1 \times e) \circ (\text{Spec}\, k \times e) \circ \Delta_{\text{Spec}\, k}$. From this fact, it becomes clear that $\mathbb{L} \otimes \mathbb{T} = \mathbb{1}$. In addition, the natural projection together with the inclusion maps $X \times \text{Spec}\, k \longrightarrow X$ and $X \longrightarrow X \times \text{Spec}\, k$ produce (through their graphs) an isomorphism $(X, p, m) \otimes \mathbb{T} \cong (X, p, m+1)$. It follows that $(X, p, m) \otimes \mathbb{L} \cong (X, p, m-1)$.

Definition 1.1.6 (Direct sum of motives)**.** Given $M = (X, p, m)$ and $N = (Y, q, n)$, if $m = n$, then we define the direct sum of them to be

$$M \oplus N = (X \bigsqcup Y, p \sqcup q, m).$$

If $m \neq n$, one can tensor with the Tate and Lefschetz motives to fall in the previous case.

A quick observation on notation: given two correspondences $p : X \longrightarrow Y, q : Z \longrightarrow W$, their product $p \times q : X \times Y \longrightarrow Z \times W$ is another correspondence. We will also write $p \times q$ as $p \otimes q$. Hence, the product of p with itself n times will be denoted as $p^{\otimes n}$.

Definition 1.1.7 (Chow rings of motives)**.** Let $M = (X, p, m)$, then p_* is a map $C^i(X) \longrightarrow C^{i+m}(X)$. We define $C^i(M) := \mathrm{Im}(p_*)$.

Similarly, if we fix a Weil cohomology theory and a work with an adequate equivalence relation finer than or equal to homological equivalence, then a projector also induces a map on cohomology and the following definition makes sense.

Definition 1.1.8 (Cohomology of motives)**.** Let $M = (X, p, m)$. Then p induces a homomorphism $p_* : H^i(X) \longrightarrow H^{i+2m}(X)$. We define $H^i(M) := \mathrm{Im}(p_*)$.

Recall that $h : \mathbf{SmProj}(k) \longrightarrow \mathbf{Mot}(k)$ is the functor sending a variety X to $(X, \Delta_X, 0)$. From the definitions it follows immediately that $C(h(X)) = C(X)$ and $H(h(X)) = H(X)$. One may ask when the gradings of $H(X)$ and $C(X)$ lift to motives, i.e., whether there exist motives $h^i(X) = (X, \Delta^i_X, 0)$ such that $h(X) = \oplus_i h^i(X) = (X, \sum \Delta^i_X = \Delta_X, 0)$ and the equalities $H(h^i(X)) = H^i(X)$ and $C(h^i(X)) = C^i(X)$ hold. At the moment, this stands as a conjecture, but many interesting cases are known, for example, for abelian varieties. Later we will define the symmetric power of a motive, and one can show that there are isomorphisms $\mathrm{Sym}^j h^1(A) \cong h^j(A)$ for every j. Those isomorphisms, after realization to cohomology, turn into the well-known relation $H^j(A) = \bigwedge^j H^1(A)$ between the cohomology groups of an abelian variety.

Now that we are familiar with the general structure of the category of motives, we shall say something about the images of projectors in this category. Recall that a morphism of motives is given by a correspondence of the form q Fix a motive $M = (X, p, m)$ and an endomorphism $f = pf'p : M \longrightarrow M$, where $f' \in \mathrm{Corr}^0(X, X)$. Also assume f is a projector in the category of motives, so $f \circ f = f$. The construction of $\mathbf{Mot}(k)$ ensures that the image of f certainly exists. The usual construction of the Karoubian completion also gives an explicit description for $\mathrm{Im}(f)$: it is given by (X, f, m). To see this, consider the morphisms of motives

$$f \circ f' \circ p : M \longrightarrow (X, f, m) \quad \text{and} \quad p \circ f' \circ f : (X, f, m) \longrightarrow M.$$

Their composition is

$$\begin{aligned}(pf'f)(ff'p) &= pf'(pf'p)(pf'p)f'p \\ &= (pf'p)f'ppf'(pf'p) \\ &= pf'ppf'ppf'ppf'p \\ &= (pf'p)(pf'p)(pf'p)(pf'p \\ &= f \circ f \circ f \circ f \\ &= f.\end{aligned}$$

Above, we used multiple times that p and f are projectors, so $p = p \circ p$ and $f = f \circ f$. From this calculation, we conclude that the maps $ff'p$ and $pf'f$ give a splitting of the projector f. A similar argument shows that this splitting is universal, so (X, f, m) is, in fact, $\mathrm{Im}(f)$.

The upshot of this argument is that the image of a projector $f : (X, p, m) \longrightarrow (X, p, m)$ is (X, f, m). Another consequence of the argument above is that $\mathrm{Im}(f)$ is a summand of M.

We finish this section with a proposition that will be very handy for later calculations.

Proposition 1.1.9 (Lieberman's identity)**.** Let $f : X \longrightarrow Y$, $\alpha : X \longrightarrow X'$, and $\beta : Y \longrightarrow Y'$ be correspondences. Then

$$(\alpha \times \beta)_*(f) = \beta \circ f \circ \alpha^{\mathrm{T}}.$$

Proof. We compute

$$(\alpha \times \beta)_*(f) = (\pi_{X' \times Y'})_* \left[(\alpha \times \beta) \cdot \pi^*_{X \times Y}(f)\right],$$

which is equal to

$$(\pi_{X' \times Y'})_* \left[(\alpha^{\mathrm{T}} \times \beta) \cdot (X' \times f \times Y')\right]$$

after switching X and X' on the product $X \times X' \times Y \times Y'$. Notice that now the projection $\pi_{X' \times Y'}$ is from $X' \times X \times Y \times Y'$. We can further rewrite the expression above as

$$(\pi_{X' \times Y'})_* \left[(\alpha^{\mathrm{T}} \times Y \times Y') \cdot (X' \times X \times \beta) \cdot (X' \times f \times Y')\right].$$

Now, notice that $\pi_{X' \times Y'}$ can be factorized as $\pi_{X' \times Y \times Y'}$, so we can write

$$(\alpha \times \beta)_*(f) = (\pi^{X' \times Y \times Y'}_{X' \times Y})_* \left[(\pi_{X' \times Y \times Y'})_* \left((\alpha^{\mathrm{T}} \times Y \times Y') \cdot \pi^*_{X' \times Y \times Y'}(X' \times \beta) \cdot (X' \times f \times Y')\right)\right].$$

Applying the projection formula gives

$$(\alpha \times \beta)_*(f) = (\pi^{X'\times Y\times Y'}_{X'\times Y})_* \left[(X' \times \beta) \cdot (\pi_{X'\times Y\times Y'})_* \left((\alpha^{\mathrm{T}} \times Y \times Y') \cdot (X' \times f \times Y')\right)\right]. \tag{1.1}$$

But $\pi_{X'\times Y\times Y'} = \pi^{X'\times X\times Y}_{X'\times Y} \times \mathrm{id}_{Y'}$, so

$$(\pi_{X'\times Y\times Y'})_* \left((\alpha^{\mathrm{T}} \times Y \times Y') \cdot (X' \times f \times Y')\right) = (\pi^{X'\times X\times Y}_{X'\times Y})_* \left((\alpha^{\mathrm{T}} \times Y) \cdot (X \times f)\right) \times Y',$$

which, by definition, is equal to $(f \circ \alpha^{\mathrm{T}}) \times Y'$.

Pluggin it in Equation 1.1, we get

$$(\alpha \times \beta)_*(f) = (\pi^{X'\times Y\times Y'}_{X'\times Y})_* \left[(X' \times \beta) \cdot \left((f \circ \alpha^{\mathrm{T}}) \times Y'\right)\right],$$

which is just $\beta \circ f \circ \alpha^{\mathrm{T}}$.

□

1.2 The standard conjectures

Grothendieck stated four conjectures that today are known as the **standard conjectures**. These conjectures have huge implications on the theory of motives and are deeply related with other important conjectures in algebraic geometry. We shall dedicate some time to introduce these conjectures, as well as describe how they connect to each other and further problems in algebraic geometry. We will present a list with four conjectures: Conjecture **A** (non-degeneracy), Conjecture **B** (Lefschetz type), Conjecture **C** (Kunneth type), Conjecture **D** (numerical = equivalence), and Conjecture Hdg (Hodge type). However, they will not be presented in this order. For the rest of this section, we fix a Weil cohomology theory.

Consider a variety X of dimension n and the cycle class of its diagonal $\gamma(\Delta_X) \in H^{2n}(X \times X)$. It decomposes, by the Kunneth decomposition, into components

$$\Delta^i_X \in H^{2n-i}(X) \otimes H^i(X).$$

Conjecture C(X) (Kunneth conjecture)**.** The components Δ^i_X are algebraic. That is, $\Delta^i_X = \gamma(\Delta_i)$ for some cycles $\Delta_i \in C^n(X \times X)$.

For a projective variety X and $\gamma(P) \in H^2(X)$ the cycle class of a (general) hyperplane section, we can define the Lefschetz operator

$$\begin{aligned} L:\ H^i(X) &\longrightarrow H^{i+2}(X) \\ \alpha &\longmapsto \alpha \cup \gamma(P). \end{aligned}$$

By definition, L is induced by an algebraic cycle. Writing d for the dimension of X, the Hard Lefschetz Theorem says that $L^j : H^{d-j}(X) \longrightarrow H^{d+j}(X)$ is an isomorphism for every $0 \leq j \leq d$. We define the map $\Lambda : H^j(X) \longrightarrow H^{j-2}(X)$ to be the composition

$$H^j(X) \xrightarrow{L^{d-j}} H^{2d-j}(X) \xrightarrow{L} H^{2d-j+2}(X) \xrightarrow{(L^{d-j+2})^{-1}} H^{j-2}(X), \quad \text{for } 0 \leq j \leq d$$

and

$$H^{2d-j}(X) \xrightarrow{(L^{d-j})^{-1}} H^j(X) \xrightarrow{L} H^{j+2}(X) \xrightarrow{L^{d-j-2}} H^{2d-j-2}(X) \quad \text{for } 0 \leq j \leq d.$$

Conjecture B(X) (Lefschetz-type conjecture)**.** The map Λ is induced by an algebraic cycle. This is to say that there is an algebraic cycle $\lambda \in \mathrm{CH}(X \times X)$ such that $\Lambda(\alpha) = (\pi_2)_*[\gamma(\lambda) \cup \pi_1^*(\alpha)]$, where π_1 and π_2 are the two projections $X \times X \longrightarrow X$.

Now, consider a variety X of dimension n and the $(n-2i+1)$-th iteration of the Lefschetz operator

$$L^{n-2i+1} : H^i(X) \longrightarrow H^{2n-2i+2}(X).$$

We write $A^i(X) := \mathrm{Im}(\gamma)$ for the algebraic cohomology classes. Now, define the i-th **primitive algebraic classes** of X to be

$$A^i_{\text{prim}}(X) := A^i(X) \cap \ker(L^{n-2i+1}) := \mathrm{Im}(\gamma) \cap \ker(L^{n-2i+1}).$$

Conjecture Hdg(X) (Hodge-type conjecture)**.** For $i \leq n/2$, the following pairing is positive-definite:

$$\begin{aligned} A^i_{\text{prim}}(X) \times A^i_{\text{prim}}(X) &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto (-1)^i \operatorname{Tr}(L^{n-2i}(\alpha) \cup \beta) \end{aligned}$$

Conjecture A(X). The pairing $A^i(X) \times A^{n-i}(X) \longrightarrow \mathbb{Q}$ is non-degenerate.

Conjecture D(X). Homological and numerical equivalence of cycles in X coincide.

We remark on the presence of a certain notational device we have been using above. Notice that we have been naming the conjectures as if they were functions of X. For example, the

Kunneth conjecture is referred to as the **Conjecture C(X)**. This allows us to talk about whether a certain conjecture holds for a specific variety, enabling us to use sentences such as "$C(\mathbb{P}^n)$ holds".

Below, we show a diagram showing the logical relations between conjectures $A(X)$, $B(X)$, $C(X)$, $D(X)$, and $\mathrm{Hdg}(X)$. An implication is represented by an arrow.

$$\begin{tikzcd}
& \mathrm{Hdg}(X)+B(X) \arrow[dl] \arrow[d] & \mathrm{Hdg}(X)+A(X) \arrow[dl] \arrow[ddl] \\
B(X) \arrow[d] \arrow[dr] & D(X) \arrow[d] & \\
C(X) & A(X) &
\end{tikzcd}$$

A less straightforward relation is that

$$A(X) \text{ for all } X \iff B(X) \text{ for all } X.$$

If we are dealing with a variety X for which we know $\mathrm{Hdg}(X)$ holds, e.g. when the field has characteristics zero, then the diagram simplifies to

$$\begin{tikzcd}
& B(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
C(X) & D(X) \arrow[r, leftrightarrow] & A(X).
\end{tikzcd}$$

If we are working over $\mathbb{C}$, the Hodge conjecture, denoted by $\mathrm{Hodge}(X)$ plays a role too:

$$\begin{tikzcd}
& \mathrm{Hodge}(X) \arrow[d] & \\
& B(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
C(X) & D(X) \arrow[r, leftrightarrow] & A(X).
\end{tikzcd}$$

For a more general case of varieties of a field of characteristics zero, Tate conjecture, represented by $\mathrm{Tate}(X)$ also implies the standard conjectures:

$$\begin{tikzcd}
& \mathrm{Tate}(X) \arrow[d] & \\
& B(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
C(X) & D(X) \arrow[r, leftrightarrow] & A(X).
\end{tikzcd}$$

In a general field of nonzero characteristics the Tate conjecture does not necessarily imples conjecture $\mathrm{Hdg}(X)$, but it still implies all other standard conjectures.

It would be difficult to paint a picture of the full collective knowledge we have on the truth value of these conjectures, but we can make an effort to write down at least some known cases.

1. Conjecture $B(X)$:
   - projective spaces;
   - grassmanians;
   - curves;
   - surfaces [Kle67];
   - abelian varieties [Kle67].
2. Conjecture $C(X)$:
   - projective spaces [Kle67];
   - grassmanians [Kle67];
   - curves ($B(X) \implies C(X)$);
   - surfaces ($B(X) \implies C(X)$);
   - abelian varieties ($B(X) \implies C(X)$);
   - flag varieties [Kle67];
   - varieties over a finite field [KM74].
3. Conjecture $D(X)$:
   - abelian varieties over fields of characteristic zero [Lie68];
4. Conjecture $\mathrm{Hdg}(X)$:
   - surfaces [Gro58];
   - varieties over a field of characteristics zero (follows from Hodge theory).

If valid, these conjectures would have important consequences on the theory of motives, as well as other topics of major interest in algebraic geometry. For instance, they were originally though of as a natural path to prove the Weil conjectures (even though the standard conjectures remain open and the Weil conjectures have been proven).

As an illustration of the impact of the standard conjectures in the theory of motives, consider the category $\mathbf{Mot}_\sim(k)$. One of the expectations is that there should be a category with nice properties such that every Weil cohomology factors through it. $\mathbf{Mot}_\sim(k)$ was constructed with the intention of being such a category. But many properties of $\mathbf{Mot}_\sim(k)$ depend on the choice of the equivalence relation $\sim$. For example, we need $\sim$ to be at least as fine as homological equivalence for $h_\sim : \mathbf{SmProj} \longrightarrow \mathbf{Mot}$ to factorize all Weil cohomology theories. Simultaneously, it is a theorem of Jannsen [Jan92] that $\mathbf{Mot}_\sim(k)$ is abelian and semisimple if and only if $\sim$ is numerical equivalence. Thus, conjecture D guarantees that there exists at least one adequate equivalence relation (namely, numerical equivalence), for which $\mathbf{Mot}_\sim(k)$ can be semisimple, abelian and factorize all Weil cohomology theories.

As a separate remark, we should mention that Jannsen's theorem is much more recent than the standard conjectures. In reality, the statement that $\mathbf{Mot}_{\mathrm{num}}(k)$ is abelian semisimple would be a consequence of conjectures $\mathrm{Hdg}(X)$ and $B(X)$ for every $X \in \mathbf{SmProj}(k)$. It just so happens, as with the Weil conjectures, that $\mathrm{Hdg}(X)$ and $B(X)$ are still open and Jannsen found another way to prove that $\mathbf{Mot}_{\mathrm{num}}(k)$ is abelian semisimple.

In some sense, the Hodge and Tate conjectures are stronger versions of the standard conjectures. Besides having the standard conjectures as consequences, they are all about the algebraicity of cohomology classes. In fact there are situations where the standard conjectures imply the Hodge and Tate conjectures (at least for some classes of varieties). For instance, we have the following theorems

Theorem 1.2.1 ([Yve04])**.** If conjecture $B(X)$ holds for every $X \in \mathbf{SmProj}(\mathbb{C})$, then the Hodge conjecture holds for abelian varieties over $\mathbb{C}$.

Theorem 1.2.2 ([Yve04])**.** If conjecture $B(X)$ holds for every $X \in \mathbf{SmProj}(\mathbb{F}_p)$, then the Tate conjecture for abelian varieties over $\mathbb{F}_p$.

Chapter 2

The dimension of motives

2.1 Kimura dimension and its properties

We introduce the concept of Kimura dimension of a motive and prove some facts about it. These ideas are built on the representation theory of the symmetric group.

In view of the correspondence between complex irreducible representations of the symmetric group S_n and partitions of n, for each partition $\lambda \vdash n$, write W_λ for the representation associated to λ (e.g., via a $\mathbb{C}[S_n]$-module structure or a map $\rho_\lambda : S_n \longrightarrow \mathrm{GL}_n(W_\lambda)$). Given a representation $\rho_\lambda : S_n \longrightarrow \mathrm{GL}_n(W_\lambda)$, its character is the morphism $\chi_\lambda : S_n \longrightarrow \mathbb{C}$ that maps $g \in S_n$ to $\chi_\lambda(g) = \mathrm{Tr}(\rho_\lambda(g))$. The idempotents

$$e_\lambda := \frac{\dim W_\lambda}{n!} \sum_{g \in S_n} \overline{\chi_\lambda(g)} \cdot g$$

generate $\mathbb{C}[S_n]$. The Artin-Wedderburn theorem then takes the form

$$\mathbb{C}[S_n] \cong \bigoplus_\lambda \mathrm{End}(W_\lambda), \quad \mathrm{End}(W_\lambda) = \mathbb{C}[S_n] \cdot e_\lambda.$$

If one takes $\lambda = (n)$, i.e., the trivial representation, then $e_{(n)} = \frac{1}{n!} \sum_{g \in S_n} g$, while the sign representation $\lambda = (1, \ldots, 1)$ gives $\frac{1}{n!} \sum_{g \in S_n} \mathrm{sgn}(g) \cdot g$. If $\mathcal{C}$ is a symmetric monoidal category, the braiding isomorphism gives an action of S_n on $X^{\otimes n}$ for any $X \in \mathcal{C}$. When $\mathcal{C}$ is also $\mathbb{Q}$-linear, one can linearly extend the action of S_n to $\mathbb{Q}[S_n]$. A formal consequence is that given two elements $r, s \in \mathbb{Q}[S_n]$ and the morphisms $r, s : X \longrightarrow X$ they determine, the composition $X \xrightarrow{r} X \xrightarrow{s} X$ is equal to the morphism determined by $s \cdot r$. In particular, for any partition $\lambda \vdash n$, the idempotent $e_\lambda \in \mathbb{Q}[S_n]$ induces a projector in $\mathcal{C}$. If $\mathcal{C}$ is further pseudo-abelian, then the image of the endomorphism induced by e_λ exists.

This described construction works in any $\mathbb{Q}$-linear pseudo-abelian symmetric monoidal category $\mathcal{C}$. As a result, for each partition $\lambda \vdash n$, we get a functor $\mathbb{T}_\lambda : \mathcal{C} \longrightarrow \mathcal{C}$. In particular, it can be done for the category of motives, we discuss now.

For any variety X, the symmetric group S_n acts on X^n by permutation of coordinates. For each $g \in S_n$, we have a morphism $X^n \longrightarrow X^n$ and the associated correspondence given by its graph $\Gamma_g(X) : X^n \longrightarrow X^n$. For $r = \sum_{g \in S_n} r(g) \cdot g$ an element of $\mathbb{Q}[S_n] \subset \mathbb{C}[S_n]$, we define

$$\Gamma_r(X) := \sum_{g \in S_n} r(g)\Gamma_g(X).$$

In this way, every element of $\mathbb{Q}[S_n]$ yields a correspondence $\Gamma_r(X) = \sum_{g \in S_n} r(g)\Gamma_g(X)$ from X^n to itself. The formal consequence $\Gamma_r(X) \circ \Gamma_s(X) = \Gamma_{r \cdot s}(X)$ we mentioned before has a concrete proof in the world of motives:

$$\begin{aligned}
\Gamma_r(X) \circ \Gamma_s(X) &= [\pi_{1,3}]_* \left[\left(\sum_{g \in S_n} r(g)\Gamma_g(X) \times X^n \right) \cdot \left(X^n \times \sum_{h \in S_n} s(h)\Gamma_h(X) \right) \right] \\
&= \sum_{g \in S_n} \sum_{h \in S_n} r(g)s(h) \cdot [\pi_{1,3}]_* \left[\left(\Gamma_g(X) \times X^n \right) \cdot \left(X^n \times \Gamma_h(X) \right) \right] \\
&= \sum_{g \in S_n} \sum_{h \in S_n} r(g)s(h) \cdot \Gamma_g(X) \circ \Gamma_h(X) \\
&= \sum_{g \in S_n} \sum_{h \in S_n} r(g)s(h) \cdot \Gamma_{g \cdot h}(X) \\
&= \sum_{b \in S_n} \left(\sum_{\substack{g,h \in S_n \\ gh = b}} r(g)s(h) \right) \Gamma_b(X) \\
&= \Gamma_{r \cdot s}(X).
\end{aligned}$$

If $M = (X, p, m)$ is a motive, then $\Gamma_g(X)$ is a morphism $M^{\otimes n} \longrightarrow M^{\otimes n}$. In particular, $\Gamma_g(X) \circ p^{\otimes n} = p^{\otimes n} \circ \Gamma_g(X)$.

Lemma 2.1.1. Fix a motive $M = (X, p, m)$. Then

1. $\Gamma_r(X) \circ p^{\otimes n} = p^{\otimes n} \circ \Gamma_r(X)$ for any $r \in \mathbb{Q}[S_n]$;
2. $(\Gamma_{e_\lambda}(X) \circ p^{\otimes n}) \circ (\Gamma_{e_\mu}(X) \circ p^{\otimes n}) = 0$ for all partitions $\lambda \neq \mu$;
3. $\sum_\lambda \Gamma_{e_\lambda}(X) = \Gamma_1$ and $(X, \Delta_X, 0) = \bigoplus_\lambda (X, \Gamma_\lambda, 0)$.

Proof. 1. This follows from the linearity of composition:

$$\left(\sum_{g \in S_n} r(g)\Gamma_g(X) \right) \circ p^{\otimes n} = \sum_{g \in S_n} r(g) \left(\Gamma_g(X) \circ p^{\otimes n} \right)$$

$$= \sum_{g \in S_n} r(g) \left(p^{\otimes n} \circ \Gamma_g(X) \right) = p^{\otimes n} \circ \left(\sum_{g \in S_n} r(g)\Gamma_g(X) \right);$$

2. This is a consequence of the fact that the e_λ are orthogonal:

$$\begin{aligned}&\left(\Gamma_{e_\lambda}(X)\circ p^{\otimes n}\right)\circ\left(\Gamma_{e_\mu}(X)\circ p^{\otimes n}\right)\\ &=\Gamma_{e_\lambda}(X)\circ\Gamma_{e_\mu}(X)\circ p^{\otimes n}\circ p^{\otimes n}\\ &=\Gamma_{e_\lambda\cdot e_\mu}(X)\circ p^{\otimes n};\end{aligned}$$

3. Similarly,

$$\sum_\lambda \Gamma_{e_\lambda}(X)=\Gamma_{(\sum_\lambda e_\lambda)}=\Gamma_1.$$

□

Finally, for all $\lambda\vdash n$, the correspondences $\Gamma_{e_\lambda}(X)$ are projectors on M and we can consider their image $\mathrm{Im}(\Gamma_{e_\lambda}(X))$, which is by definition $\mathbb{T}_\lambda M$. Recalling how one computes images of projectors, we have the following definition.

Definition 2.1.2. Given a motive $M=(X,p,m)$ and a partition λ of n, we define

$$\mathbb{T}_\lambda M=(X^n,\Gamma_{e_\lambda}\circ p^{\otimes n},nm)=\mathrm{Im}(\Gamma_{e_\lambda}(X)).$$

For $\lambda=(1,\dots,1)$ or $\lambda=(n)$, we write

$$\bigwedge^n M:=\mathbb{T}_{(1,\dots,1)}M \ \text{ and } \ \mathrm{Sym}^n M:=\mathbb{T}_{(n)}M.$$

Note that since the e_λ generate the group ring $\mathbb{Q}[S_n]$, we get $M^{\otimes n}=\bigoplus_{\lambda\vdash n}\mathrm{Im}(\Gamma_{e_\lambda}(X))$.

Definition 2.1.3 (Kimura dimension of a motive)**.** Let $M\in\mathbf{Mot}(k)$ be a motive.

- M is **even** if $\bigwedge^n M=0$ for all n sufficiently large. The maximal n for which $\bigwedge^n M\neq 0$ is called the **even dimension** of M.
- M is **odd** if $\mathrm{Sym}^n M=0$ for all n sufficiently large. The maximal n for which $\mathrm{Sym}^n M\neq 0$ is called the **odd dimension** of M.
- M is **finite-dimensional** if it can be written as $M=M_+\oplus M_-$ where M_+ is even and M_- is odd. In this case, the **dimension** of M is the sum of the odd and even dimensions of M_- and M_+, respectively.

This definition makes sense in the context of general pseudo-abelian symmetric monoidal categories. In addition, if M is finite-dimensional, it may be the case that there are multiple ways to decompose M into a sum $M_+\oplus M_-$ of even and odd motives. So, it may not be clear that $\dim(M)$ is well defined, but we will see shortly that such a decomposition is unique up to isomorphism.

Since finite dimensionality comes in two flavors: even or odd, we will often talk about the parity of a finite-dimensional motive, in the same vein as one talks about the parity of an integer. For example, two motives have the same parity if they are both even or both odd.

The condition $\operatorname{Sym}^n M = (X^n, \Gamma_{e_{(n)}} \circ p^{\otimes n}, nm) = 0$, and analogously for $\bigwedge^n M$, means that $\Gamma_{e_{(n)}} \circ p^{\otimes n} \sim 0$. Since $S_n \subset S_{n+1}$, we can write $e_{(n+1)}$ as a product $e_{(n+1)} = r \cdot e_{(n)}$ for some r. Hence,

$$\Gamma_{e_{(n+1)}} = \Gamma_r \circ \Gamma_{e_{(n)}},$$

which means that whenever $\operatorname{Sym}^n M$ is zero, so is $\operatorname{Sym}^{n+1} M$, and the analogous implication holds for $\bigwedge^n M$. In synthesis, if $\operatorname{Sym}^n M = 0$ then the odd dimension of M is less than n, and the analogous statement holds for the even dimension.

Given a motive $M = (X, p, m)$, we may decompose $H(M)$ into an odd part $H^{\mathrm{odd}}(M) := \bigoplus_k H^{2k+1}(M)$ and an even one $H^{\mathrm{even}}(M) := \bigoplus_k H^{2k}(M)$. Then, we have the following proposition.

Proposition 2.1.4. For any motive M, we have

$$H(\operatorname{Sym}^n M) = \bigoplus_{i+j=n} \operatorname{Sym}^i H^{\mathrm{even}}(M) \otimes \bigwedge^j H^{\mathrm{odd}}(M)$$

and

$$H\left(\bigwedge^n M\right) = \bigoplus_{i+j=n} \bigwedge^i H^{\mathrm{even}}(M) \otimes \operatorname{Sym}^j H^{\mathrm{odd}}(M).$$

We mentioned before that in the case of motives of abelian varieties, there is a decomposition $h(A) = \bigoplus_i h^i(A)$ and isomorphism $\operatorname{Sym}^i(h^1(A)) \cong h^i(A)$. With the formulas above, we can, for example, calculate

$$H^n(A) = H(h^n(A)) \cong H(\operatorname{Sym}^n h^1(A) = \bigoplus_{i+j=n} \operatorname{Sym}^i H^{\mathrm{even}}(h^1(A)) \otimes \bigwedge^j H^{\mathrm{odd}}(h^1(A)).$$

But $H^i(h^1(A)) = 0$ whenever $i \neq 1$, so $H^{\mathrm{even}}(h^1(A)) = 0$. Thus, the only non-vanishing term is when $i = 0$, so $j = n$ and we get

$$H^n(A) = H(h^n(A)) = \bigwedge^n H^{\mathrm{odd}}(h^1(A)) = \bigwedge^n H^1(A).$$

Lemma 2.1.5. Given two motives $M, N \in \mathbf{Mot}(k)$, we have the formulas

$$\mathrm{Sym}^n M \oplus N) = \bigoplus_{i+j=n} \mathrm{Sym}^i M \otimes \mathrm{Sym}^j N$$

and

$$\bigwedge^n (M \oplus N) = \bigoplus_{i+j=n} \bigwedge^i M \otimes \bigwedge^j N.$$

Proof. Let $M = (X, p, m)$ and $N = (Y, q, n)$. We will only show the first one and for the case $m = n$, as the proof of the second is analogous and the the general situation where $m \neq n$ follows from twisting by Tate and Lefschetz motives. By definition,

$$\mathrm{Sym}^n (M \oplus N) = \Big((X \sqcup Y)^n, \Gamma_{e_{(n)}}(X \sqcup Y) \circ (p \sqcup q)^{\otimes n}, nm \Big).$$

Now, we introduce the following notation. For $I \subset [n] = \{1, 2, \ldots, n\}$ and $J = [n] \setminus I$, we write $X^I \times Y^J$ for the product of $|I|$ copies of X and $|J|$ copies of Y where the copies of X appear in the positions indexed by the elements of I and the copies of Y appear in the positions indexed by the elements of J. Then we have decompositions

$$(X \sqcup Y)^n = \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} X^I \times Y^J,$$

and

$$(p \sqcup q)^{\otimes n} = \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} p^{\otimes I} \times q^{\otimes J}.$$

Consequently,

$$\mathrm{Sym}^n (M \oplus N) = \left(\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} X^I \times Y^J,\ \Gamma_{e_{(n)}}(X \sqcup Y) \circ \left[\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} p^{\otimes I} \times q^{\otimes J} \right],\ nm \right).$$

If we write $\Gamma_{e_{(n)}}(X \sqcup Y)$ as $\frac{1}{n!} \sum_{g \in S_n} \Gamma_g(X \sqcup Y)$, we can study $\Gamma_g(X \sqcup Y)$ for each $g \in S_n$. For a permutation $g \in S_n$, notice that $\Gamma_g(X \sqcup Y)$ can be written as the following intersection product:

$$\Delta_{1,g(1)}(X \sqcup Y) \cdot \Delta_{2,g(2)}(X \sqcup Y) \cdots \Delta_{n,g(n)}(X \sqcup Y),$$

where $\Delta_{i,j}(X \sqcup Y)$ stands for the cycle on $(X \sqcup Y)^n$ given by $\{(x_1, \ldots, x_n) \in (X \sqcup Y)^n : x_i = x_j\}$.

Hence, we have the composition

$$\Gamma_g(X \sqcup Y) \circ \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} p^{\otimes I} \times q^{\otimes J} = \Delta_{1,g(1)}(X \sqcup Y) \cdot \Delta_{2,g(2)}(X \sqcup Y) \cdots \Delta_{n,g(n)}(X \sqcup Y) \circ \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} p^{\otimes I} \times q^{\otimes J}$$

$$= \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} \left[\prod_{i \in I} \Delta_{i,g(i)}(X) \circ p^I \right] \times \left[\prod_{j \in J} \Delta_{j,g(j)}(Y) \circ p^J \right].$$

This way, we can write $\mathrm{Sym}^n(M \oplus N)$ as

$$\mathrm{Sym}^n(M \oplus N) = \left(\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} X^I \times Y^J, \frac{1}{n!} \sum_{g \in S_n} \Gamma_g(X \sqcup Y) \circ \left[\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} p^{\otimes I} \times q^{\otimes J} \right], nm \right)$$

$$= \left(\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} X^I \times Y^J, \frac{1}{n!} \sum_{g \in S_n} \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} \left[\prod_{i \in I} \Delta_{i,g(i)}(X) \circ p^I \right] \times \left[\prod_{j \in J} \Delta_{j,g(j)}(Y) \circ p^J \right], nm \right)$$

$$= \left(\bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} X^I \times Y^J, \ \frac{1}{n!} \bigsqcup_{\substack{I \subset [n] \\ J = [n] \setminus I}} \sum_{\substack{h \in S_I \\ h' \in S_J}} \left[\Gamma_h(X) \circ p^I \right] \times \left[\Gamma_{h'}(Y) \circ p^J \right], nm \right). \qquad (2.1)$$

Now, we can use the fact that for any two partitions I, J and I', J' of $[n]$ with $|I| = |I'|$ we have isomorphisms

$$(X^I \times Y^J, p^I \times q^J, nm) \cong (X^{I'} \times Y^{J'}, p^{I'} \times q^{J'}, nm),$$

to rewrite Equation 2.1 as a union indexed by the possible sizes of I and J, accounting for how many partitions there are for each given size. Fixed $0 < i, j < n$ with $i + j = n$, there are $\frac{n!}{i!j!}$ partitions I, J of $[n]$ such that $|I| = i$ and $|J| = j$, so Equation 2.1 becomes

$$= \left(\bigsqcup_{i+j=n} X^i \times Y^j, \ \frac{n!}{i!j!} \cdot \frac{1}{n!} \bigsqcup_{i+j=n} \sum_{g \in S_i} \sum_{h \in S_j} \left[\Gamma_g(X) \circ \ p^{\otimes i} \times \Gamma_h(Y) \circ \ q^{\otimes j} \right], \ (i+j)m \right)$$

$$= \left(\bigsqcup_{i+j=n} X^i \times Y^j, \ \bigsqcup_{i+j=n} \left(\frac{1}{i!} \sum_{g \in S_i} \Gamma_g(X) \circ \ p^{\otimes i} \right) \times \left(\frac{1}{j!} \sum_{h \in S_j} \Gamma_h(Y) \circ \ q^{\otimes j} \right), \ (i+j)m \right)$$

$$= \bigoplus_{i+j=n} \left(X^i \times Y^j, \Gamma_{e_{(i)}}(X) \circ \ p^{\otimes i} \times \Gamma_{e_{(j)}}(Y) \circ \ q^{\otimes j}, \ (i+j)m \right)$$

$$= \bigoplus_{i+j=n} \left(X^i, \Gamma_{e_{(i)}}(X) \circ \ p^{\otimes i}, \ im \right) \otimes \left(Y^j, \Gamma_{e_{(j)}}(Y) \circ \ q^{\otimes j}, \ jm \right)$$

$$= \bigoplus_{i+j=n} \mathrm{Sym}^i M \otimes \mathrm{Sym}^j N.$$

□

Theorem 2.1.6. Given two motives $M, N \in \mathbf{Mot}(k)$, the following are true.

1. M and N are both even if and only if $M \oplus N$ is even;

2. M and N are both odd if and only if $M \oplus N$ is odd;

3. if M and N are finite dimensional, then so is $M \oplus N$. In addition, $\dim(M \oplus N) \leq \dim(M) + \dim(N)$.

Proof. 1. Suppose $\bigwedge^{k+1} M = \bigwedge^{l+1} N = 0$, with k and l the even dimensions of M and N, respectively. The last lemma says

$$\bigwedge^{k+l+1}(M \oplus N) = \bigoplus_{i+j=k+l+1} \bigwedge^{i} M \otimes \bigwedge^{j} N = 0$$

since any pair i, j summing $i + j = k + l + 1$ will have $i > k$ or $j > l$, so the even dimension of $M \oplus N$ is less than or equal to $k + l$.

Conversely, if $\bigwedge^{k}(M \oplus N) = 0$ for some k, every summand $\bigwedge^{i} M \otimes \bigwedge^{j} N$ must vanish, including when $i = k$ or $j = k$.

2. The argument is the same as above.

3. Suppose $M = M_+ \oplus M_-$ and $N = N_+ \oplus N_-$ with M_+, N_+ even and M_-, N_- odd. In this scenario, the two items we proved above show that $M_+ \oplus N_+$ is even and $M_- \oplus N_-$ is odd, so

$$M \oplus N = (M_+ \oplus N_+) \oplus (M_- \oplus N_-)$$

presents $M \oplus N$ as a decomposition into an even and an odd part. In addition, the dimension of $M \oplus N$ is, by definition, equal to $\dim(M_+ \oplus N_+) + \dim(M_- \oplus N_-)$, while

$$\dim(M_+ \oplus N_+) \leq \dim(M_+) + \dim(N_+),$$
$$\dim(M_- \oplus N_-) \leq \dim(M_-) + \dim(N_-).$$

Hence, we obtain

$$\dim(M \oplus N) \leq \dim(M_+) + \dim(M_-) + \dim(N_+) + \dim(M_-) = \dim(M) + \dim(N).$$

□

We remark that the converse of the third item of the theorem above also holds (Corollary 2.1.20): as we will see, if $M \oplus N$ is finite-dimensional, then both M and N are also finite-dimensional. It is just that we did not develop the technology to show this yet.

Recall that a *Young Tableau* on a Young diagram is simply a way to fill the squares of the Young diagram with non-repeating numbers from 1 up to the number of squares. Given a Young tableau T on the Young diagram associated with $\lambda \vdash n$, the symmetric group S_n acts on the diagram by permuting the squares according to their numbers. We obtain two subgroups of S_n:

$$R_\lambda(T) := \{g \in S_n : \text{ each row of } T \text{ is invariant by } g\},$$
$$C_\lambda(T) := \{g \in S_n : \text{ each column of } T \text{ is invariant by } g\}.$$

We also get the following three elements of $\mathbb{Q}[S_n]$:

$$a_\lambda(T) := \sum_{g \in R_\lambda(T)} g,$$
$$b_\lambda(T) := \sum_{g \in C_\lambda(T)} \operatorname{sgn}(g) g,$$
$$c_\lambda(T) := a_\lambda(T) b_\lambda(T).$$

These are idempotents, defining projectors $X^{\otimes n} \longrightarrow X^{\otimes n}$ for any $X \in \mathcal{C}$ in a pseudo-abelian symmetric monoidal category. Furthermore, the images of $a_\lambda(T)$ and $b_\lambda(T)$ do not depend on the choice of tableau T and they are equal to

$$a_\lambda(M) = \operatorname{Sym}^{\lambda_1} M \otimes \cdots \operatorname{Sym}^{\lambda_s} M$$
$$b_\lambda(M) = \bigwedge^{\lambda'_1} M \otimes \cdots \bigwedge^{\lambda'_r} M,$$

where $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ is the conjugate partition of λ.

The assignment $X \in \mathcal{C} \longmapsto \operatorname{Im}(c_\lambda)$, where $c_\lambda : X^{\otimes n} \longrightarrow X^{\otimes n}$, defines a functor known as the **Schur functor** associated with λ and is denoted by S_λ. One can then define $X \in \mathcal{C}$ to be **Schur finite** if $S_\lambda(X) = 0$ for some λ. A Kimura finite-dimensional object is always Schur finite, but the converse does not hold. For example, O'Sullivan constructed a Schur finite-dimensional motive that is not Kimura finite-dimensional. We will not work with the notion of Schur finite-dimensional motive here, but it has important relations with conjectures about mixed motives.

Lemma 2.1.7. Given $n \geq k$, a motive M, and a partition $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$, we have

1. if $\operatorname{Sym}^{k+1} M = 0$ and $\lambda_1 > k$, then $\mathbb{T}_\lambda M = 0$;

2. if $\bigwedge^{k+1} M = 0$ and $s \leq k+1$, then $\mathbb{T}_\lambda M = 0$.

There is a more general form of this lemma, where, for example, in the first item, one requires that $\lambda_i > k$ for some i.

Proof. We only show the first part as the proof for the second part is analogous. If $M = (X, p, m)$, then $\mathbb{T}_\lambda M = (M^n, \Gamma_{e_\lambda} \circ p^{\otimes n}, nm)$. Fixing a tableau T for λ, we write

$$a_\lambda := a_\lambda(T),$$
$$b_\lambda := b_\lambda(T),$$
$$c_\lambda := c_\lambda(T) = a_\lambda b_\lambda.$$

As $e_\lambda = r \cdot c_\lambda$ for some $r \in \mathbb{Q}[S_n]$, we have $\Gamma_{e_\lambda}(X) = \Gamma_r(X) \circ \Gamma_{a_\lambda}(X) \circ \Gamma_{b_\lambda}(X)$. We argue that $\Gamma_{a_\lambda}(X) = 0$. The image of a_λ is

$$\mathrm{Im}(a_\lambda) = \mathrm{Sym}^{\lambda_1} M \otimes \cdots \otimes \mathrm{Sym}^{\lambda_s} M.$$

In the case $\lambda_1 > k$, we have $\mathrm{Sym}^{\lambda_1} M = 0$, so $\mathbb{T}_\lambda M = 0$. □

Theorem 2.1.8. Let M, N be two motives. Then,

1. if M, N have the same parity, then $M \otimes N$ is even;

2. if M and N have different parity, then $M \otimes N$ is odd.

In both cases, $\dim(M \otimes N) \leq \dim(M)\dim(N)$.

Proof. 1. Suppose $M = (X, p, m), N = (Y, q, n)$ are both odd and let d and l be their respective dimensions (the proof for the even case is analogous). We will show that $\bigwedge^{dl+1}(M \otimes N) = 0$. First of all,

$$(M \otimes N)^{dl+1} = M^{dl+1} \otimes N^{dl+1} = \bigoplus_{\lambda, \mu \vdash dl+1} \mathrm{Im}(\Gamma_{e_\lambda}(X)) \otimes \mathrm{Im}(\Gamma_{e_\mu}(Y)).$$

Thus we can compute $\bigwedge^{dl+1}(M \otimes N)$ by calculating the image of $\Gamma_{e_{(1,\ldots,1)}}(X \times Y)$ on the direct sum above, which gives us

$$\begin{aligned}
&\Gamma_{e_{(1,\dots,1)}}(X\times Y)\left(\bigoplus_{\lambda,\mu\vdash dl+1}\mathrm{Im}(\Gamma_{e_\lambda}(X))\otimes \mathrm{Im}(\Gamma_{e_\mu}(Y))\right)\\
&=\bigoplus_{\lambda,\mu\vdash dl+1}\Gamma_{e_{(1,\dots,1)}}(X\times Y)\left[\mathrm{Im}(\Gamma_{e_\lambda}(X))\otimes \mathrm{Im}(\Gamma_{e_\mu}(Y))\right]\\
&=\bigoplus_{\lambda,\mu\vdash dl+1}\mathrm{Im}\left[\Gamma_{e_{(1,\dots,1)}}(X\times Y)\circ(\Gamma_{e_\lambda}(X)\otimes\Gamma_{e_\mu}(Y))\right].
\end{aligned}$$

To continue, we use that for any two partitions λ,μ of a number L, the following relation holds:

$$e_{(1,\dots,1)}\cdot e_\lambda\otimes e_\mu=\begin{cases}e_{(1,\dots,1)} \text{ if } \mu=\lambda',\\ 0 \text{ otherwise.}\end{cases}$$

Above, λ' means the conjugate partition of λ. As a consequence, the summands

$$\Gamma_{e_{(1,\dots,1)}}(X\times Y)\circ\mathrm{Im}(\Gamma_{e_\lambda}(X))\otimes \mathrm{Im}(\Gamma_{e_\mu}(Y))$$

vanish every time $\lambda'\neq\mu$. So we only need to show that the product of the form $\mathbb{T}_\lambda M\otimes \mathbb{T}_{\lambda'}N$ are trivial. Write $\lambda=(\lambda_1,\dots,\lambda_s)$ and $\lambda'=(\lambda'_1,\dots,\lambda'_r)$. If $\lambda_i>d$ for some i, then Lemma 2.1.7 implies $\mathbb{T}_\lambda M=0$ and we are done. If $\lambda_i\leq d$ for all i, then s must be at least $l+1$, so $\lambda'_1>l$ and we conclude that $\mathbb{T}_{\lambda'}N=0$.

2. Now, consider two motives $M=(X,p,m)$ and $N=(Y,q,n)$, where M is even of dimension d and N is odd of dimension l (if the parities were reversed, we just change the roles of M and N). Similar calculations as the ones done for the first item give

$$\mathrm{Sym}^{dl+1}(M\otimes N)=\bigoplus_{\lambda,\mu\vdash dl+1}\mathrm{Im}\left[\Gamma_{e_{(dl+1)}}(X\times Y)\circ(\Gamma_{e_\lambda}(X)\otimes\Gamma_{e_\mu}(Y))\right].$$

This time, however, the relations we use are

$$e_{(dl+1)}\cdot e_\lambda\otimes e_\mu=\begin{cases}e_{(dl+1)} \text{ if } \mu=\lambda,\\ 0 \text{ otherwise.}\end{cases}$$

So we reduced the problem to show that the summands of the form $\mathbb{T}_\lambda M\otimes\mathbb{T}_\lambda N$ are zero for every $\lambda\vdash dl+1$. If some λ_i is greater than l, then Lemma 2.1.7 implies $\mathbb{T}_\lambda N=0$. If this is not the case, then $\lambda_i\leq l$ for every i, forcing $s\geq d+1$, which means $\mathbb{T}_\lambda M = 0$.

In the respective analyzed cases, we saw that $\mathrm{Sym}^{dl+1}(M \otimes N)$ and $\bigwedge^{dl+1}(M \otimes N)$ vanish, so the dimension of $M \otimes N$ can not be greater than $dl = \dim(M)\dim(N)$. □

Corollary 2.1.9. Given two varieties $X, Y \in \mathbf{SmProj}(k)$, if the motives $h(X)$ and $h(Y)$ are finite-dimensional, then so is $h(X \times Y)$.

Definition 2.1.10 (Smash-nilpotent morphism)**.** A morphism $f : M \longrightarrow N$ between motives is said to be **smash-nilpotent** if there exists $n > 0$ such that $f^{\otimes n} = 0$.

Notice that if $\sim$ is rational equivalence, then f being a smash-nilpotent *morphism* is the same as the associated correspondence being trivial under the *smash-nilpotent equivalence relation on cycles*.

Lemma 2.1.11. If $f, g : M \longrightarrow N$ are two smash-nilpotent morphisms, then so is $f + g$ and $f - g$.

Proof. For any positive integer $n > 0$, we have

$$(f+g)^{\otimes n} = \sum_{i+j=n} \binom{n}{i} f^{\otimes i} \times g^{\otimes j}.$$

Since f, g are smash-nilpotent, for large enough n, the condition $i + j = n$ forces $f^{\otimes i}$ or $g^{\otimes j}$ to be zero. A similar argument can be made for $f - g$. □

Proposition 2.1.12. Let $f : M \longrightarrow N$ be a smash-nilpotent morphism with $f^{\otimes n} \sim 0$ and $g_i : N \longrightarrow M, 1 \leq i \leq n-1$ a sequence of morphisms. Then, $f g_{n-1} f \cdots f g_1 f = 0$.

Proof. Write $M = (X, p, m)$ and $N = (Y, q, n)$. Let $X \times Y \times X \times \cdots \times X \times Y$ be the product of n copies of X and n copies of Y. Write $\pi_{i,j}$ for the projection into the product $X \times Y$ of the i-th copy of X and the j-th copy of Y. With the occasional aid of the projection formula, one can write the composition $f g_1 f \cdots g_{n-1} f$ as

$$[\pi_{1,n}]_* \left(\pi^*_{n,n}(f) \cdot \pi^*_{n,n-1}(g_{n-1}) \cdot \pi^*_{n-1,n-1}(f) \cdots \pi^*_{2,2}(f) \cdot \pi_{2,1}(g_1) \cdot \pi^*_{1,1}(f)\right),$$

which vanishes since the intersection product between the $\pi^*_{i,i}(f)$, for $1 \leq i \leq n$, is zero by our hypothesis. □

Corollary 2.1.13. If $f : M \longrightarrow M$ is a smash-nilpotent morphism, then it is nilpotent with respect to composition. Moreover, if $n \in \mathbb{N}$ is such that $f^{\otimes n} = 0$, then $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} = 0$.

Proposition 2.1.14. A morphisms between finite-dimensional motives of different parities is smash-nilpotent.

Proof. Let $f : M = (X, p, m) \longrightarrow N = (Y, q, n)$ be a map of motives with M even and N odd (the reverse situation is analogous). Write $d = \dim(M)$ and $l = \dim(N)$. We will show that $f^{\otimes k} = 0$ for any $k > dl$.

Consider two partitions $\lambda, \mu \vdash k$ and the composition $\Gamma_{e_\mu}(Y) \circ f^{\otimes k} \circ \Gamma_{e_\lambda}(X)$. We can commute either $\Gamma_{e_\lambda}(X)$ or $\Gamma_{e_\mu}(Y)$ with $f^{\otimes k}$. We then get

$$\Gamma_{e_\mu \cdot e_\lambda}(Y) \circ f^{\otimes k} = \Gamma_{e_\mu}(Y) \circ \Gamma_{e_\lambda}(Y) \circ f^{\otimes k} = f^{\otimes k} \circ \Gamma_{e_\mu}(X) \circ \Gamma_{e_\lambda}(X) = f^{\otimes k} \circ \Gamma_{e_\mu \cdot e_\lambda}(X).$$

But $e_\mu \cdot e_\lambda$ is zero if $\mu \neq \lambda$. Thus, for the composition above to not be zero, it would need to be of the form $\Gamma_{e_\lambda}(Y) \circ f^{\otimes k} = f^{\otimes k} \circ \Gamma_{e_\lambda}(X)$. Writing $\lambda = (\lambda_1, \ldots, \lambda_s)$, Lemma 2.1.7 implies that if any $\lambda_i > l$, then $\mathbb{T}_\lambda N = 0$ (recall that N is odd), so the composition $\Gamma_{e_\lambda}(Y) \circ f^{\otimes k}$ is zero. If this is not the case, then the same lemma implies $\mathbb{T}_\lambda M = 0$ and we conclude that $f^{\otimes k} \circ \Gamma_{e_\lambda}(X) = 0$. As $\Gamma_{e_\lambda}(Y) \circ f^{\otimes k} = f^{\otimes k} \circ \Gamma_{e_\lambda}(X)$, we just showed that

$$\Gamma_{e_\lambda}(X) \circ f^{\otimes k} = 0$$

for any λ. But $\sum_\lambda e_\lambda = 1$, so $f^{\otimes k}$ must be zero. □

Corollary 2.1.15. If a motive M is both even and odd, then $M = 0$.

Proof. Write $M = (X, p, m)$ and apply the last proposition to the identity morphism $p = \mathrm{id}_M : M \longrightarrow M$. We get $\mathrm{id}_M = p = p^n = 0$, and by Corollary 2.1.13 we get that $M = 0$. □

Theorem 2.1.16. Let M be a finite-dimensional motive with $M = M_+ \oplus M_- = M'_+ \oplus M'_-$, where M_+, M'_+ are even and M_-, M'_- are odd. Then, $M_+ \cong M'_+$ and $M_- \cong M'_-$.

Proof. Write $M = (X, p, m)$ and p_+, p'_+, p_-, p'_- for the projectors associated to M_+, M'_+, M_-, M'_-, respectively. Thus, $p = p_+ + p_- = p'_+ + p'_-$. Consider the composition

$$M_+ \xrightarrow{p_+} M_+ \oplus M_- = M'_+ \oplus M'_- \xrightarrow{p'_-} M'_-,$$

which is smash-nilpotent, hence nilpotent, by Proposition 2.1.14. It means that, for some $n > 0$, $(p'_- \circ p_+)^n = 0$. We use the fact that $p'_- = p - p'_+$ and since $p : M \longrightarrow M$ is the identity morphism, we have $p \circ p_+ = p_+$, so when expanding $(p'_- \circ p_+)^n = 0$, we get

$$((p - p'_+)p_+)^n = (p_+ - p'_+ p_+)^n = p_+^n - F(p_+, p'_+)p'_+ p_+ = 0$$

for some (possibly non-commutative) polynomial F. We may rearrange this expression to obtain $p_+^n = p_+ = F(p_+, p'_+)p'_+ p_+$. The composition $p'_+ \circ p_+ : M_+ \longrightarrow M \longrightarrow M'_+$ defines a morphism of motives, and so does $F(p_+, p'_+) : M'_+ \longrightarrow M_+$. But we just saw that their composition is p_+, the identity on M_+. This way, $F(p_+, p'_+)$ is a splitting for $p'_+ \circ p_+$, which gives a factorization

$$M'_+ \cong M_+ \oplus \ker(F(p_+, p'_+)).$$

Theorem 2.1.6 implies that $\ker(F(p_+, p'_+))$ is also even and that

$$\dim(M_+) + \dim(\ker F(p_+, p'_+)) = \dim(M'_+) \geq \dim(M_+).$$

Changing the roles of p_+ and p'_+ then gives the reverse inequality, forcing $\dim \ker(F(p_+, p'_+))$ to be zero, which means $\ker(F(p_+, p'_+))$ is zero. We conclude that $M'_+ \cong M_+$. Repeating the argument for M_- and M'_- finishes the proof. □

Definition 2.1.17 (Surjective morphism)**.** A morphism of motives $f : M \longrightarrow N$ is said to be **surjective** if for all $Z \in \mathbf{SmProj}(k)$, the map

$$C(M \otimes h(Z)) \longrightarrow C(N \otimes h(Z))$$

is surjective.

Lemma 2.1.18. Let $f : M = (X, p, m) \longrightarrow N = (Y, q, n)$ be a morphism of motives, then the following are equivalent.

1. f is a surjective morphism;

2. f has a right inverse (it is a split epimorphism);

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$: using Lieberman's identity, we see that $(q \times \Delta_Y)_*(\Delta_Y) = q^T$, so that q^T is in the image of $q \times \Delta_Y$, which is the projector defining $N \times h(Y)$. In other words, $q^T \in C(N \times h(Y))$. The surjectivity of f means there is some correspondence $r \in \mathrm{Corr}^0(X, Y)$ such that

$$q^{\mathrm{T}} = (f \times \Delta_Y)_*(r) = r \circ f^{\mathrm{T}}.$$

Now, let $g = pr^{\mathrm{T}}q$, then we have

$$f \circ g = fpr^{\mathrm{T}}q = fr^{\mathrm{T}}q = q \circ q = q,$$

which is the identity of N, so g is a right inverse of f.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$: From our assumptions, there is some $g : N \longrightarrow M$ with $f \circ g = q = \mathrm{id}_N$. Passsing to Chow groups, we get $(f \times \Delta_Z)_* \circ (g \times \Delta_Z)_* = \mathrm{id}_{C(M \otimes h(Z))}$, so $(f \times \Delta_Z)_*$ is surjective. □

Theorem 2.1.19. Let $f : M \longrightarrow N$ be a surjective morphism of motives, then if M is finite-dimensional, so is N.

Proof. Suppose M is even or odd. Lemma 2.1.18 implies that f has a right inverse g, so we split M into the summands $M = N \oplus \ker(f)$, implying that N is finite-dimensional and of the same parity as M (this is Theorem 2.1.6).

Now, let us look at the case where $M = (X, p, m) = M_+ \oplus M_-$, where $M_+ = (X_+, p_+, m)$ is even and $M_- = (X_-, p_-, m)$ is odd. Write $N = (Y, q, n)$, so from Lemma 2.1.18, there is $s : Y \longrightarrow X$ such that $f \circ s = q$. From $p = p_+ + p_-$, we get

$$q = (fs)q = (fps)q = fp_+sq + fp_-sq.$$

We will write q_+ for fp_+sq and q_- for fp_-sq. These correspondences will be used to construct two projectors q'_+ and q'_- that will give us a decomposition of $N = (Y, p_+, n) \oplus (Y, p_-, n)$. First of all, observe that $q_\pm \circ q = q_\pm = q \circ q_\pm$. Also, $q_+ \circ q_-$ is nilpotent: it is equal to fp_+sqfp_-q and the composition p_+sqfp_- is (by Proposition 2.1.14) a smash-nilpotent morphism between the motives $M_- \longrightarrow M_+$, which have different parities. Finally, the smash-nilpotency of p_+sqfp_- together with Proposition 2.1.12 guarantees that $q_+ \circ q_-$ is nilpotent.

Say $(q_+ \circ q_-)^k = 0$ and define

$$q'_+ = \left[q - q_-^k\right]^k \text{ and } q'_- = q - q'_+.$$

Notice that there exists a polynomial P such that for any correspondence $t : Y \longrightarrow Y$, we have $(q - t)^k = q - P(t)t$ or, equivalently,

$$P(t)t = q - (q - t)^k. \tag{2.2}$$

Choosing $t = q_+$ and taking the k-th power of equation 2.2, we get

$$q'_+ = (q - q_-^k)^k = (q - (q - q_+)^k)^k = (P(q_+)q_+)^k = P(q_+)^k q_+^k = q_+^k P(q_+)^k, \tag{2.3}$$

and making $t = q_-^k$, we get

$$q'_- = q - q'_+ = q - (q - q_-^k)^k = P(q_-^k)q_-^k = q_-^k P(q_-^k). \tag{2.4}$$

We will also need the following observation:

$$0 = (q_+q_-)^k = q_+^k q_-^k, \tag{2.5}$$

where the first equality is restating that $q_+ \circ q_-$ is a nilpotent, which we already proved, and the second is a consequence from the fact that q_+ and $q_- = q - q_+$ commute. Now we compute $q'_+ q_+^k = (q - q'_-)q_+^k$. Substituting Equation 2.4 into that, we obtain

$$q'_+ \circ q_+^k = (q - q'_-)q_+^k = (q - P(q_-^k)q_-^k) \circ q_+^k = qq_+^k - P(q_-^k)q_-^k q_+^k,$$

where the second term vanishes because of Equation 2.5. So $q'_+ \circ q_+^k = q \circ q_+^k = q_+^k$. This allows us to conclude that q'_+ is an idempotent by the following calculation:

$$q'_+ \circ q'_+ = q'_+ q_+^k P(q_+)^k = q_+ P(q_+)^k = q'_+.$$

The first and last equalities follow from Equation 2.3, while the middle one is the fact $q'_+ \circ q_+^k = q_+^k$ that we just proved.

If we now compute

$$q'_- \circ q'_- = (q - q'_+)(q - q'_+) = q^2 - qq'_+ - q'_+ q + {q'_+}^2 = q - q'_+ - q'_+ + q'_+ = q'_-,$$

we conclude that q'_- is also an idempotent. In addition, q'_+ and q'_- are orthogonal by construction. Finally, we define the motives $N_+ = (Y, q'_+, n)$ and $N_- = (Y, q'_-, n)$. It follows that $N = N_+ \oplus N_-$. The only thing left to check is that N_+ is even and N_- is odd. If we write $t = p_+ sq$, then since $q_+ = f p_+ sq = f \circ t$, from Equation 2.3 we get

$$q'_+ = q_+^k \circ P(q_+)^k = f \circ \left(t \circ P(f \circ t)^k \right),$$

providing a right inverse for f. Lemma 2.1.18 then implies that $f : M_+ \longrightarrow N_+$ is surjective. We already showed the part of this theorem that asserts that a motive which is the target of a surjection from an even variety is also even. By an analogous argument, one can also prove that N_- is odd. □

Corollary 2.1.20. If $M \oplus N$ is a finite-dimensional motive, then so are M and N.

Proof. Just apply the theorem we have just proved to the projections $M \oplus N \longrightarrow M$ and $M \oplus N \longrightarrow N$. □

Corollary 2.1.21. Let $\phi : X \longrightarrow Y$ be a dominant morphism of varieties. If $h(X)$ is finite-dimensional, then so is $h(Y)$.

Proof. We just need to show that $\Gamma_\phi = \phi_* : h(X) \longrightarrow h(Y)$ is surjective. If ϕ is generically finite of degree r, then $\phi_* \circ \phi^* = r\Delta_Y$, so ϕ_* has a right inverse and, by Lemma 2.1.18, it is surjective. In the case ϕ is not generically finite, we take a rational multisection X' of $\phi : X \longrightarrow Y$, so $\phi|_{X'} : X' \longrightarrow Y$ is generically finite, say of degree r. Now, for any divisor $W \subset Y$ we have $\phi_*(X' \cdot \phi^*(W)) = rW$, so ϕ_* is surjective and we conclude that $h(Y)$ also is finite-dimensional. □

So far, we have been developing the theory of finite-dimensional motives. It is time to devote some energy to collect all the results we have established and start recognizing some finite-dimensional motives. Unfortunately, we will not be able to explicitly say which motives are finite-dimensional and which are not, this is still an open problem (see Conjecture 2.1). However, we can describe some classes of motives that are finite-dimensional. We start by working out some simple examples and build upon these examples with the tools we have been setting up.

Let $C \in \mathbf{SmProj}(k)$ be a smooth curve with at least one rational point $e \in C$. We can associate two motives to C:

$$h^0(C) := (C, p_0 = e \times C, 0) \quad \text{and} \quad h^2(C) := (C, p_2 = C \times e, 0).$$

Together, they induce a decomposition $h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$, where $h^1(C) := (C, p_1, 0) = (C, \Delta - p_0 - p_2, 0)$. Consider the motive $\bigwedge^2 h^0(C)$. By definition, it is equal to

$$(C^2, \Gamma_{e_{(1,1)}}(C) \circ p_0^{\otimes 2}, 0) = (C^2, \frac{1}{2}\left[\Gamma_1(C) \circ p_0^{\otimes 2} - \Gamma_\sigma(C) \circ p_0^{\otimes 2}\right], 0).$$

Computing the composition of the correspondences yields

$$\Gamma_1(C) \circ p_0^{\otimes 2} = p_0^{\otimes 2} \quad \text{and} \quad \Gamma_\sigma(C) \circ p_0^{\otimes 2} = p_0^{\otimes 2},$$

so that $\Gamma_{e_{(1,1)}}(C) \circ p_0^{\otimes 2} = 0$. Thus, the motive $h^0(C)$ is even of dimension one. The same argument can be replicated to $h^2(C)$, so we end up with the following lemma.

Lemma 2.1.22. For any smooth projective curve $C \in \mathbf{SmProj}(k)$, the motive $h^0(C) \oplus h^2(C)$ is even of dimension 2.

Lemma 2.1.23. Let $C \in \mathbf{SmProj}(k)$ be a smooth curve of genus g. The motive $h^1(C)$ is odd of dimension $2g$.

Corollary 2.1.24. The motive $h(C)$ is finite-dimensional for any $C \in \mathbf{SmProj}(k)$.

Before proving this result, we will see some immediate consequences

Corollary 2.1.25. If $X \in \mathbf{SmProj}(k)$ is dominated by a product of curves, then $h(X)$ is finite-dimensional

Proof. This follows from combining Corollary 2.1.9 with Corollary 2.1.21. □

Proposition 2.1.26. The motive $h(X)$ is finite-dimensional for the following classes of varieties $X \in \mathbf{SmProj}(k)$:

1. abelian varieties;
2. varieties with dimension less or equal to 3 that are rationally dominated by products of curves;
3. K3 surfaces with Picard number 19 or 20;
4. Hilbert schemes of points of surfaces with finite-dimensional motive;
5. Fano varieties of lines of smooth cubic threefolds and fivefold;

More generally, combining our previous results with Corollary 2.1.24 shows that the subcategory of $\mathbf{Mot}(k)$ generated by the motives of curves contains only motives of finite dimension. All the varieties on the list above have motives that lie in this category. We will not show in details this proposition but we will briefly explain why one should expect these varieties to have finite-dimensional motives.

1. Symmetric powers of curves have finite-dimensional motives, since they are quotients of products of curves. And Jacobian varieties are dominated by these symmetric powers, so they too have finite-dimensional motive. Every Abelian variety is a quotient of a Jacobian variety (see [Mil86]), so Abelian varieties have finite-dimensional motives.
2. The dominant rational map induces a surjective map in top Chow groups (over an appropriate field). This surjection allows one to write the motive of the variety as a sum of the motive of the product of curves, the motive of some other curves, and the motive of either a curve or a surface that has a finite-dimensional motive [Via17]. In any case, the result is the sum of finite-dimensional motives.
3. The motive of a surface X decomposes into a sum of six motives, all finite-dimensional but possibly one. This motive $t(X)$ is called the transcendental part of $h(X)$. If X is a K3 surface with Picard number 19 or 20, it has a Nikulin involution whose quotient is birational to Y, the Kummer surface of an abelian surface [Mor84]. Since Y has finite-dimensional motive,

the summand $t(Y)$ is also finite-dimensional. It turns out that $t(X) \cong t(Y)$, so the only summand of $h(X)$ that could fail to be finite-dimensional is finite-dimensional [Ped12].

4. One uses a stratification of the Hilbert scheme of points to give isomorphisms between its Chow groups and sums of Chow groups of symmetric powers of the surface. These induce isomorphisms on motives after the appropriate twists [CM02].

5. Given a smooth cubic hypersurface $X \subset \mathbb{P}^n$, there is a birational map from the second Hilbert scheme $X^{[2]}$ to a certain projective bundle over X [GS14]. This allows one to write the motive of $F(X)$ in terms of $h(X)$ and $h(X^{[2]})$, and $h(X^{[2]})$ is finite-dimensional when $h(X)$ is too (see [Lat17] for the details). In the case $\dim X$ is 3 or 5, the Abel-Jacobi Chow groups of X vanish, so its motive is finite-dimensional by [Via13].

It raises the question of how large can we make the subcategory of finite-dimensional motives.

Conjecture (Kimura-O'Sullivan)**.** Every Chow motive is finite-dimensional.

We will later discuss the implications this conjecture would have for the theory of motives. Right now, we turn our attention to the proof of the Lemma 2.1.23. Proposition 2.1.4 implies that

$$H(\operatorname{Sym}^{2g} h^1(C)) = \bigoplus_{i+j=2g} \operatorname{Sym}^i H^{\text{even}}(h^1(C)) \otimes \bigwedge^j H^{\text{odd}}(h^1(C)),$$

but

$$H(C) = H(h(C)) = H(h^0(C) \oplus h^1(C) \oplus h^2(C)),$$

with $H(h^0(C)) = H^0(C)$ and $H(h^2(C)) = H^2(C)$, so that $H(h^1(C)) = H^1(C)$. Thus, the even part of $H(h^1(C))$ is trivial and we get

$$H(\operatorname{Sym}^{2g} h^1(C)) = \bigwedge^{2g} H^1(C),$$

which is non-zero since C has genus g.

From this calculation, we found that if $h^1(C)$ is odd, then its dimension must be at least $2g$. We will now show that $\operatorname{Sym}^{2g+1} h^1(C) = 0$. Define the projector

$$\alpha_n = \frac{1}{n!} \sum_{g \in S_n} \Gamma_g(C) \circ p_1^{\otimes n}$$

of C^n. By definition, $\operatorname{Sym}^n h^1(C) = (C^n, \alpha_n, 0)$, so it suffices to show that $\alpha_{2g+1} = 0$.

We will do so in two main steps. First, we define another projector β_n and show that $\alpha_n = 0$ if and only if $\beta_n = 0$. After that, we will prove that β_{2g+1} is zero by pushing forward certain computations in $\mathrm{CH}(S^{2g+1}C)$ by the projective bundle map $S^{2g+1}C \longrightarrow J(C)$ and performing these calculations in $\mathrm{CH}(J(C))$.

Write S^nC for the symmetric power of C, i.e., the quotient of C^n by the permutation action of S_n. Also denote by ϕ_n the projection $\phi_n : C^n \longrightarrow S^nC$. We define the correspondence β_n as

$$\beta_n = \frac{1}{n!}(\phi_n)_* \circ \alpha_n \circ \phi_n^* = \frac{1}{n!}(\phi_n \times \phi_n)_*(\alpha_n).$$

The first equality above is the definition of β_n, while the second is Lieberman's identity.

Lemma 2.1.27. For all $n > 0$, we have

1. β_n is a projector of S^nC;
2. $\alpha_n = \frac{1}{n!}\phi_n^* \circ \beta_n \circ (\phi_n)_*$;
3. $\mathrm{Sym}^n h^1(C) \cong (S^nC, \beta_n, 0)$.

Proof. 1. Using the fact that $\phi_n^* \circ (\phi_n)_* = (\Gamma_{\phi_n})^{\mathrm{T}} \circ \Gamma_{\phi_n} = n! \cdot \Gamma_{e_{(n)}}(C)$, we have

$$\beta_n \circ \beta_n = \frac{1}{n!}(\phi_n)_*\alpha_n\phi_n^*\frac{1}{n!}(\phi_n)_*\alpha_n\phi_n^* = \frac{1}{n!}(\phi_n)_*\alpha^2\phi_n^* = \frac{1}{n!}(\phi_n)_*\alpha \circ \phi_n^* = \beta_n.$$

2.

$$\frac{1}{n!}\phi_n^* \circ \beta_n \circ (\phi_n)_* = \frac{1}{n!}\phi_n^* \circ \frac{1}{n!}(\phi_n)_* \circ \alpha_n \circ \phi_n^* \circ (\phi_n)_* = \Delta_{C^n} \circ \alpha_n \circ \Delta_{C^n} = \alpha_n.$$

3. We have the two morphisms of motives

$$\frac{1}{n!}\beta_n \circ (\phi_n)_* \circ \alpha_n : \mathrm{Sym}^n h^1(C) \longrightarrow (S^nC, \beta_n, 0)$$

and

$$\alpha_n \circ \phi_n^* \circ \beta_n : (S^nC, \beta_n, 0) \longrightarrow \mathrm{Sym}^n h^1(C),$$

that compose to

$$\frac{1}{n!}\beta_n \circ (\phi_n)_*\alpha_n\alpha_n\phi_n^*\beta_n = \beta_n \underbrace{\frac{1}{n!}(\phi_n)_*\alpha_n\phi_n^*}_{\beta_n}\beta_n = \beta_n$$

and

$$\alpha_n\phi_n^*\beta_n\frac{1}{n!}\beta_n \circ (\phi_n)_*\alpha_n = \alpha_n \underbrace{\frac{1}{n!}\phi_n^*\beta_n \circ (\phi_n)_*}_{\alpha_n}\alpha_n = \alpha_n.$$

Notice that above we have used both the first and second items of this lemma, which we have just proved. Recall that α_n and β_n are the identities of $\mathrm{Sym}^n h^1(C)$ and $(S^nC, \beta_n, 0)$, respectively, so the calculations above show that they are isomorphic.

□

The interesting part of the above lemma is really the last item, and the first two are only stepping stones to get there. The isomorphism between $\mathrm{Sym}^n h^1(C)$ and $(S^nC, \beta_n, 0)$ means that in order to show that $\alpha_n = 0$, it is enough to demonstrate that $\beta_n = 0$.

Consider the two projection maps $\pi_1, \pi_2 : S^{2g+1}C \times S^{2g+1}C \longrightarrow S^{2g+1}C$ and the Abel-Jacobi map $\psi : S^{2g+1}C \longrightarrow J(C)$, which is a projective bundle with fibers $\mathbb{P}^{g+1}$. We can consider the relative tautological bundle over the fibers $\mathbb{P}^{g+1}$ of the map $S^{2g+1}C \longrightarrow J(C)$. Write ξ for the class in the Chow ring of the divisor associated to this tautological bundle. We then have a projective bundle formula on Chow rings:

$$\mathrm{CH}(S^{2g+1}C) = \mathrm{CH}(J(C))[1, \xi, \xi^2, \ldots, \xi^{g+1}].$$

So $\beta_{2g+1} \in \mathrm{CH}(S^{2g+1}C \times S^{2g+1}C)$ can be written in terms of pullbacks of the ξ^j by the projections π_1, π_2 and pullbacks of some cycles in $\mathrm{CH}(J(C))$ by the compositions $\phi \circ \pi_1$ and $\phi \circ \pi_2$. Explicitly, there are cycles $a_{ij} \in \mathrm{CH}(J(C) \times J(C))$ such that

$$\beta_{2g+1} = \sum_{i,j=0}^{g+1} (\psi \circ \pi_1 \times \psi \circ \pi_2)^*(a_{ij}) \cdot \pi_1^*(\xi^i) \cdot \pi_2^*(\xi^j).$$

We will show that the a_{ij} above are all trivial. To do so, we will need the following lemma.

Lemma 2.1.28. The intersection product $\pi_1^*(\xi) \cdot \beta_{2g+1}$ vanishes.

Proof. Let us state a helpful fact: given $e \in C$, for $n > 2g - 2$, the inclusion

$$\begin{array}{ccc} S^nC & \xrightarrow{i} & S^{n+1}C \\ (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n, e) \end{array}$$

has the property that $i_*(S^nC)$ is in the same class as the relative tautological bundle on $S^{n+1}C$ (remember $S^{n+1}C \longrightarrow J(C)$ is a projective bundle). In the particular case where $n = 2g$, we obtain $\pi_1^*(\xi) = \pi_1^*(S^{2g}C \times e) = S^{2g}C \times e \times S^{2g+1}C$.

Using this new piece of information together with Lieberman's identity, we can write the product $\pi_1^*(\xi)\cdot\beta_{2g+1}$ as

$$\frac{1}{(2g+1)!}\pi_1^*(\xi)\cdot(\phi_{2g+1}\times\phi_{2g+1})_*(\alpha_{2g+1}) = \frac{1}{(2g+1)!}S^{2g}C\times e\times S^{2g+1}C\cdot(\phi_{2g+1}\times\phi_{2g+1})_*(\alpha_{2g+1}).$$

Now, the projection formula yields

$$\frac{1}{(2g+1)!}(\phi_{2g+1}\times\phi_{2g+1})_*\left[(\phi_{2g+1}\times\phi_{2g+1})^*(S^{2g}C\times e\times S^{2g+1}C)\cdot\alpha_{2g+1}\right].$$

In the pullback $(\phi_{2g+1}\times\phi_{2g+1})^*(S^{2g}C\times e\times S^{2g+1}C)$, we are viewing $S^{2g}C\times e$ as a cycle in $\mathrm{CH}(S^{2g+1}C)$, so that $\phi_{2g+1}^*(S^{2g}C\times e)=\sum_j C^j\times e\times C^{2g-j}$. Hence,

$$\pi_1^*(\xi)\cdot\beta_{2g+1} = \frac{1}{(2g+1)!}(\phi_{2g+1}\times\phi_{2g+1})_*\left[\sum_j C^j\times e\times C^{2g-j}\times C^{2g+1}\cdot\alpha_{2g+1}\right].$$

We will check that for each j, the sum $\sum_j C^j\times e\times C^{2g-j}\times C^{2g+1}\cdot\alpha_{2g+1}$ is zero. To see that, recall that α_{2g+1} is defined as $\alpha_{2g+1}=\frac{1}{n!}\sum_{h\in S_{2g+1}}\Gamma_h(C)\circ p_1^{\otimes 2g+1}$, where p_1 is the projector $\Delta_C - e\times C - C\times e$. So, it is enough to prove that $C^j\times e\times C^{2g-j}\times C^{2g+1}\cdot\left(\Gamma_h(C)\circ p_1^{\otimes 2g+1}\right)=0$ for each $1\leq j\leq 2g$ and $h\in S_{2g+1}$.

By acting on the factors of $\sum_j C^j\times e\times\times C^{2g-j}\times C^{2g+1}$ and $\Gamma_h(C)\circ p_1^{\otimes 2g+1}$ by an appropriate element of S_{2g+1}, we reduce the problem to show that $e\times C^{2g}\times C^{2g+1}\cdot\left(\Gamma_h(C)\circ p_1^{\otimes 2g+1}\right)$ is zero for every $h\in S_{2g+1}$, which follows from the computation

$$(e\times C)\cdot p_1 = (e\times C)\cdot\Delta_C - (e\times C)\cdot(e\times C) - (e\times C)\cdot(C\times e) = 0$$

and the fact that we are working with an adequate equivalence relation. □

Proof of Lemma 2.1.23. Recall that

$$\beta_{2g+1} = \sum_{i,j=0}^{g+1}(\psi\circ\pi_1\times\psi\circ\pi_2)^*(a_{ij})\cdot\pi_1^*(\xi^i)\cdot\pi_2^*(\xi^j),$$

then we can do an inductive argument as follows. First, notice that since β_{2g+1} is a cycle of codimension $2g+1$ in $S^nC\times S^nC$ and the fibers of $\psi\circ\pi_1\times\psi\circ\pi_2$ have dimension $4g+2$, a codimension counting shows that $a_{g+1,g+1}=0$. Now, suppose we have shown that $a_{ij}=0$ for all $g+1\geq i,j>k$. Then multiply β_{2g+1} by $\pi_1^*(\xi^{g+1-k})\cdot\pi_2^*(\xi^{g-k})$ and push it forward by

$\psi \circ \pi_1 \times \psi \circ \pi_2$, so we get

$$\begin{aligned}(\psi \circ \pi_1 \times \psi \circ \pi_2)_* &\left(\beta_{2g+1} \cdot \pi_1^*(\xi^{g+1-k}) \cdot \pi_2^*(\xi^{g-k})\right) \\ &= (\psi \circ \pi_1 \times \psi \circ \pi_2)_* \left[\sum_{i,j=0}^{k} (\psi \circ \pi_1 \times \psi \circ \pi_2)^*(a_{ij}) \cdot \pi_1^*(\xi^{g+1-k+i}) \cdot \pi_2^*(\xi^{g-k+j})\right] \\ &= \sum_{i,j=0}^{k} a_{ij} \cdot (\psi \circ \pi_1 \times \psi \circ \pi_2)_* \left[\pi_1^*(\xi^{g+1-k+i}) \cdot \pi_2^*(\xi^{g-k+j})\right],\end{aligned}$$

where we used the projection formula to get get from the second to the last line. But $\pi_1^*(\xi^l)$ is non-zero only if $l = g+1$, so

$$(\psi \circ \pi_1 \times \psi \circ \pi_2)_* \left[\pi_1^*(\xi^{g+1-k+i}) \cdot \pi_2^*(\xi^{g-k+j})\right]$$

can only be non-zero when $i = k$ and $j = k+1$. When that happens,

$$(\psi \circ \pi_1 \times \psi \circ \pi_2)_* \left[\pi_1^*(\xi^{g+1-k+i}) \cdot \pi_2^*(\xi^{g-k+j})\right] = 1,$$

so all we are left with is a_{ij}. Consequently,

$$a_{ij} = (\psi \circ \pi_1 \times \psi \circ \pi_2)_* \left(\beta_{2g+1} \cdot \pi_1^*(\xi^{g+1-k}) \cdot \pi_2^*(\xi^{g-k})\right).$$

Finally, Lemma 2.1.28 implies that a_{ij} is zero. Changing the roles of i and j allows one to show the result for $i, j \geq k$ with not both equal to k. To get the result for $i = j = k$, one can now replicate the argument but multiply β_{2g+1} by $\pi_1^*(\xi^{g+1-k}) \cdot \pi_2^*(\xi^{g+1-k})$ before pushing it forward by $\psi \circ \pi_1 \times \psi \circ \pi_2$. That concludes the inductive step. □

2.2 Finite dimensionality in the theory of motives

In this last section, we will compile relations of the Kimura-O'Sullivan conjecture with other problems in the theory of motives and related topics. First of all, let us remember what the conjecture says.

Conjecture (Kimura-O'Sullivan)**.** Every Chow motive is Kimura finite-dimensional.

Let us list some simple consequences of this conjecture. In the list below, we work with the category of motives built from rational equivalence.

- If $M = (X, p, m)$ is a finite-dimensional motive with p is numerically trivial, then $M = 0$;

- any morphism $f : M \longrightarrow M$ that is numerically trivial (as a correspondence on $M \times M$) is also nilpotent.
- if M is finite-dimensional, its dimension is equal to the dimension of its cohomology: $\dim(M) = \dim(H(M))$;
- let X be a variety whose cycle class maps (with coefficients in the base field k) $\mathrm{CH}^i(X)_k \longrightarrow H^{2i}(X)$ is a surjection for all i. If M is finite-dimensional, then the cycle class maps are also injective;

Let us talk about more elaborated repercussions now. When we defined the cohomology of a motive, we mentioned the problem of lifting the grading on $H(h(X)) = H(X)$ to motives $h^i(X)$ such that $h(X) = \bigoplus_i h^i(X)$. This idea is embodied by the following definition.

Definition 2.2.1 (Chow-Kunneth decomposition)**.** Let $X \in \mathbf{SmProj}(k)$ be a smooth projective variety and write Δ_X^i for the components of $\gamma(\Delta_X) \in H(X \times X)$ under the Kunneth decomposition. We say that X admits a **Chow-Kunneth decomposition** if there are projectors p_i on X such that

1. $\sum_i p_i = \Delta_X$;
2. the p_i are orthogonal idempotents;
3. $\gamma(p_i) = \Delta_X^i$.

Conjecture CK(X). Every variety admits a Chow-Kunneth decomposition.

Notice that, in particular, Conjecture $CK(X)$ implies Conjecture $C(X)$. The converse holds whenever $h(X)$ is finite-dimensional.

Theorem 2.2.2. Let $X \in \mathbf{SmProj}(k)$ be a smooth projective variety. If $h(X)$ is a finite-dimensional motive and Conjecture $C(X)$ holds, then Conjecture $CK(X)$ also holds.

During our discussion about the finite-dimensionality of curves, we showed, for example, that Conjecture $CK(X)$ holds when $X = C$ is a curve. More generally, it is also true for surfaces and abelian varieties. Conjecture $CK(X)$ is only one piece of the so-called Murre conjecture, which prescribes a certain filtration on Chow groups.

Conjecture (Murre's conjecture)**.** Let $X \in \mathbf{SmProj}(k)$ be a smooth projective variety. We say that **Murre's conjecture** holds for X if Conjecture $CK(X)$ holds and the filtrations defined by

$$F^k \mathrm{CH}^j(X) = \ker(p_{2j}) \cap \ker(p_{2j-1}) \cap \cdots \cap \ker(p_{2j+1-k}) \subset \mathrm{CH}^j(X)$$

satisfy the conditions

1. $F^1 \operatorname{CH}^j(X) = \operatorname{CH}(X)_{\text{hom}} := \{z \in \operatorname{CH}^j(X) : Z \sim_{\text{hom}} 0\}$;

2. for any k and j, the group $F^k \operatorname{CH}^j(X)$ does not depend on the choice of the projectors p_i giving the Chow-Kunneth decomposition of X.

Interestingly, Murre's conjecture for every $X \in \mathbf{SmProj}(k)$ is equivalent to the Bloch-Beilinson conjectures on the filtrations of Chow groups. In addition, if they are true, Bloch-Beilinson's filtration coincides with the filtration defined above.

Another interesting relation between finite-dimensionality of motives and other open problems in algebraic geometry comes when we look at varieties defined over finite fields. In this case, Kahn showed the following theorems.

Theorem 2.2.3 ([Kah03])**.** Given a variety $X \in \mathbf{SmProj}(\mathbb{F}_q)$ whose associated motive $h(X)$ is finite-dimensional, Tate's conjecture implies the Beilinson conjecture for X.

Theorem 2.2.4 ([Kah03])**.** The following are equivalent:

1. Tate's conjecture is true for every smooth projective variety and Kimura-O'Sullivan's conjecture holds;

2. Tate's conjecture is true for every abelian variety and $\mathbf{Mot}(\mathbb{F}_q)$ is equal to the subcategory generated by curves.

The second item implies, but is stronger than, Kimura-O'Sullivan conjecture. We have already seen that the subcategory generated by curves only contains motives of finite dimension. So if this subcategory contains all motives, of course Kimura-O'Sullivan conjecture holds, but in principle: it could be the case that every motive is finite-dimensional and some of them are not generated by motives of abelian varieties. Finally, we have the following theorem relating Tate's conjecture, Kimura finiteness and K-theory.

Theorem 2.2.5 ([Kah03])**.** Let $X \in \mathbf{SmProj}(X)$ be a smooth projective variety such that $h(X)$ is finite-dimensional. If Tate's conjecture holds for X, then all higher algebraic K-groups of X with rational coefficients vanish: $K_k(X) \otimes \mathbb{Q} = 0$ for $k > 0$.

Changing the focus now (not restricting ourselves to finite fields anymore), we talked about conjecture $D(X)$ when discussing the standard conjectures. There is a stronger conjecture that says numerical equivalence coincides with smash-nilpotent equivalence (recall that a cycle $Z \in \operatorname{CH}(X)$ is smash-nilpotent if $Z^n \in \operatorname{CH}(X^n)$ is trivial for some $n > 0$).

Conjecture (Voevodsky)**.** For any $X \in \mathbf{SmProj}(k)$, cycles on X up to numerical equivalence and smash-nilpotent equivalence coincide.

Homological equivalence sits between smash-nilpotent equivalence and numerical equivalence, so Voevodsky's conjecture implies conjecture $D(X)$ for all X. We also have the following theorem.

Theorem 2.2.6. Voevodsky's conjecture implies Kimura-O'Sullivan's conjecture.

The conjecture of Kimura-O'Sullivan, or more precisely the notion of finite-dimensionality, also relates to Bloch conjectures in the case of surfaces.

Theorem 2.2.7. Given a smooth projective surface S, if $h(S) \in \mathbf{Mot}_{\text{rat}}(S)$ is finite-dimensional, then Bloch's conjecture holds for S.

Going on with the applications of the theory of finite-dimensional motives, we can bring up the problem of classifying motivic zeta functions that are rational. The Grothendieck ring of varieties $K_0(\mathbf{Var})$ is defined to be the free abelian group over isomorphism classes of varieties quotiented by the equivalence relation of *scissors congruence*, that is,

$$[X] \sim [X \setminus U] + [U] \quad \text{if } U \text{ is an open subvariety of } X.$$

We can endow $K_0(\mathbf{Var})$ with a ring structure by declaring $[X] \cdot [Y] = [X \times Y]$.

The **motivic zeta function** is the power series

$$Z([X], t) = 1 + [X] \cdot t + [S^2(X)] \cdot t^2 + [S^3(X)] \cdot t^3 + \cdots .$$

For example, when X is a smooth projective curve of genus g, its zeta function becomes

$$Z([X], t) = \frac{P(t)}{(1-t)(1-[\mathbb{A}^n] \cdot t)},$$

where $P(t)$ is a polynomial of degree $2g$. In particular, $Z(X, t)$ is a rational function. It was conjectured by Kapranov that this is the case for every variety and later Larsen and Lunts proved (see [LL03]) this is not the case.

Now, observe that $K_0(\mathbf{Mot}(k))$ is a commutative monoid, so we may apply the usual construction of the Grothendieck group (in this case a ring due to the tensor product in $\mathbf{Mot}(k)$), yielding the ring $K_0(\mathbf{Mot}(k))$. Concretely, its elements are classes of motives up to the relation

$$[M] \sim [N] + [L] \quad \text{if} \quad M = N \oplus L.$$

Then, one may ask whether the zeta function on motives

$$Z([M],t) = 1 + [M]\cdot t + [\mathrm{Sym}^2 M]\cdot t^2 + [\mathrm{Sym}^3 M]\cdot t^3 + \cdots$$

is rational. We don't know, for example, if all motives have rational zeta function, but it is the case for finite-dimensional motives.

Theorem 2.2.8. If M is finite-dimensional, then $Z(M,t)$ is rational.

We would like to finish this section with some comments on how things work with mixed motives. For mixed motives, it is not true that every motive is Kimura finite-dimensional. In reality, Schur finiteness is better suited for this setting. Recall that a (mixed) motive M is Schur finite-dimensional if there λ such that $S_\lambda(M) = 0$. Kimura finite-dimensional objects are always Schur finite. The reason Schur finiteness is more appropriate is because it behaves nicely with respect to the triangulated structure on the category of mixed motives. More precissely, if three mixed motives M, N, L fit into a distinguished triangle

$$M \longrightarrow N \longrightarrow L \longrightarrow \Sigma M$$

in Voevodsky's triangulated category, and two of M, N, L are Schur finite-dimensional, then also is the third. Schur finiteness relates to important open question on the theory of mixed motives. For instance, the work of Ayoub on the conservativity conjecture (see [Ayo07]) links it to conditions of Schur finiteness of certain mixed motives.

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