Elliptic cohomology, genera and the index of Dirac operators in loop spaces

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1 Introduction

In broad terms, elliptic cohomology theories are a class of generalized cohomology theories associated with elliptic curves. They are oriented cohomology theories, which provide us, between other things, with the notion of a genus (plural is genera). Genera are homomorphisms from some cobordism ring, and the genera arising from elliptic cohomology theories are the elliptic genera.

Historically, however, elliptic genera came before elliptic cohomology and a solid framework to properly unify all these "elliptic things" required some years of effort and some remarkable insights. These notes intend to discuss some aspects of this process, with a focus on some observations made by Witten concerning elliptic genera and index theory in loop spaces, as well as how it impacted the understanding of elliptic cohomology and the next developments in the field.

We take, despite the historical ordering of the facts discussed here, a rather modern approach to the topic. That means we draw intuition and directions from more contemporary approaches, placing historical remarks when relevant.

2 Elliptic cohomology theories

We start with a simple statement: "Elliptic cohomology theories are generalized cohomology theories". Of course they are more than that. Not all cohomology theories are elliptic, but this statement at least provide some context for what to expect.

Perhaps a good question to ask right now could be "there are many generalized cohomology theories around, why should I care about elliptic ones instead of any other?".

The answer for that comes in two pieces. First, you should also care about other cohomology theories, not just elliptic ones. Maybe not all of them are interesting, but there is a fair amount of interesting ones. Second, we have multiple sources source of interest for elliptic cohomology theories that could support the position that they are among the interesting ones.

A primary reason is the fact that they correspond to the second level in the chromatic filtration. In short terms, there is a correspondence between formal group laws over suitable rings to (complex oriented) cohomology theories and the height filtration of formal group laws induces a similar filtration on cohomology theories, the *chromatic filtration*. This is the content of chromatic homotopy theory. For example, in the 0-th level of the chromatic filtration, one may find ordinary cohomology theory, while the first level contains complex K-theory. The existence of a classifying ring of formal group laws yields a "universal" complex-oriented cohomology theory, which turns out to be complex cobordism.

Elliptic cohomology theories live in the second level of the chromatic filtration, so it is the "next simple thing" after ordinary cohomology theory and complex K-theory (and some Morava K-theories). Another reason to care about Elliptic cohomology may be its recent appearance in derived algebraic geometry, although this is a bit distant from the topics we will be discussing here. A third reason lies in its applicability in mathematical physics, particularly in string theory, which was a major motive for Witten to even think about these things.

We shall start by properly defining what is an elliptic cohomology theory. First, we establish some necessary concepts.

Recall that a cohomology theory E is multiplicative if it allows for a graded ring structure $E_\bullet$. In particular, it has a unit living in E^0 and the spectrum representing E promotes to a ring spectrum.

Definition 1. *Given a multiplicative generalized cohomology theory E, consider the inclusion*

$$\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$$

and the induced map on reduced second cohomology

$$\tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(\mathbb{C}P^2).$$

A ***complex orientation*** *for E is an element $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ which is mapped to the unity $1 \in \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2) = \tilde{E}^2(\Sigma^2 S^0) \cong \tilde{E}^0(S^0) = E^0(S^0, \{*\}) = E^0(*)$. In such situation, E is said to be a* ***complex oriented cohomology theory****.*

We will, for the rest of this text, omit the adjective "complex" from "complex oriented", as we will not deal with or care about any other kind of orientation.

Remark. *Not every (multiplicative) generalized cohomology theory admits a complex orientation. The ones that do may receive the adjective "orientable" and become orientable cohomology theory. Saying that a cohomology theory is oriented implicitly assumes a previous choice of orientation.*

From a complex orientation, one can build a **genus** for the cohomology theory E. By *genus* here, we mean a ring homomorphism from the complex cobordism ring $\mathrm{MU}_\bullet$ to $E_\bullet$, that is, a ring spectrum map $\mathrm{MU} \to E$. A precise statement would be

Theorem. *Ring spectrum maps* $\mathrm{MU} \to E$ *are in bijection with complex orientations for* E.

The way one would go about proving it is by showing the universality of cobordism as a complex-oriented cohomology theory. More precisely, one builds a universal complex orientation on MU and for each ring spectrum map $\mathrm{MU} \to E$, pushforward the orientation on MU to obtain a complex orientation on E. The important fact for us here is that complex orientations naturally give us some notion of genera.

The last ingredient we need is a small discussion about group laws. We define a formal group law below.

Definition 2. *A* ***formal group law*** *over a ring* R *is a formal power series* $F \in R[[x, y]]$ *with coefficients in* R *satisfying:*

1. $F(x, 0) = x$;
2. $F(x, y) = F(y, x)$;
3. $F(x, F(y, z)) = F(F(x, y), z)$.

Some words about the first property may be useful to get some intuition. The first property stipulates that F has no term in degree 0 and the terms of degree higher than 1 always have both factors x and y. Simple examples of formal power series include

Example 1. $F(x, y) = x + y$.

Example 2. $F(x, y) = x + y + c \cdot xy$ *for some coefficient* $c \in R$.

We can define maps between formal group laws:

Definition 3. *If* F *and* G *are formal group laws over* R*, a* ***homomorphism*** *from* F *to* G *is a power series* f *with coefficients in* R *such that*

$$f(F(x, y)) = G(f(x), f(y)).$$

We say f *is an isomorphism if it has an inverse (in the power series sense) which is also a homomorphism.*

A natural source of formal group laws is elliptic curves. Elliptic curves have a group structure and we can derive a formal group law from it. One can consider the Taylor expansion around the origin of the addition law (in coordinates), which yields a power series. This is the *formal group law associated with an elliptic curve.*

What we need to keep in mind now is that there is a functorial way to assign cohomology theories to formal group laws. This is via the Landweber exact functor $\mathrm{MU}_\bullet \otimes_L (-)$. The important piece of information is that given a formal group law over a graded ring R, the formal group law induces an algebra structure on R over the Lazard ring L so we tensorize $\mathrm{MU}_\bullet$ and R over L. This establishes a correspondence between complex-oriented cohomology theories and suitable formal group laws. We are now ready to define elliptic cohomology.

Definition 4. *An **Elliptic cohomology theory** is a generalized multiplicative cohomology theory E such that*

- *E is even and periodic, that is, $E^n(*) = 0$ for n odd and there is some invertible element $\beta \in E^2(*)$. This forces E to be complex oriented (so it has a formal group law associated);*
- *$E^0(*) \cong R$ for a commutative ring R;*
- *There is an elliptic curve C over R such that the formal law associated to E is isomorphic to the one associated with C.*

The definition above says that the "generalized" E-Chern classes built on E (the Conner-Floyd Chern class) behave as sum on an elliptic curve. A more precise way to phrase this is to say that if F is the formal group law associated to E and c_1 is the first Conner-Floyd Chern class of E, then for any line bundles L_1, L_2, we have $c_1(L_1 \oplus L_2) = F(c_1(L_1), c_1(L_2))$. Interestingly, if F is like in Example 1, then E is ordinary cohomology theory. If F is the same as in Example 2 with the constant $c = 1$, then E is K-theory.

3 Elliptic genera

Another thing called *genus* was going around before the formalization of the genus associated with a complex oriented cohomology theories. A **genus** in this sense is simply a ring homomorphism $\mathrm{MSO}_\bullet \to R$ to some ring R. Unpacking this idea we may organize the definition in the following way.

Definition 5. *An R-valued **genus** is a ring homomorphism $\phi : \mathrm{MSO}_\bullet \to R$, that is, a rule assigning to each manifold M, an element $\phi(M) \in R$ such that for any two manifolds M, N, we have*

- *$\phi(M \sqcup N) = \phi(M) + \phi(N)$;*
- *$\phi(M \times N) = \phi(M) \times \phi(N)$;*
- *If M and N are orientably cobordant, then $\phi(M) = \phi(N)$.*

Remark. *One could modify the definition of genus above by changing the ring $\mathrm{MSO}_\bullet$ to any other cobordism theory, yielding different notions of genus that relate to different adjectives for cohomology theories. Indeed, some would call our definition "oriented genus" instead of simply "genus".*

From Thom's calculation of the rationalization of $\mathrm{MSO}_\bullet$, we have $\mathrm{MSO}_\bullet \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \dots]$, so rationally, one can represent a genus by a polynomial series using the classes of the projective spaces $\mathbb{C}P^{2n}$. An advantage of rationalizing the oriented cobordism ring is that we get corresponding power series depending only on the classes of projective spaces.

To any formal group law F over a commutative $\mathbb{Q}$-algebra R, there is a logarithm associated. This is a power series $\log_F$ such that $\log_F(F(x,y)) = \log_F(x) + \log_F(y)$. So we have a power series associated to any genus ϕ, which is determined by the values of ϕ on the projective spaces. This power series, which we will call $\log_\phi$, is given by applying ϕ to the coefficients of the logarithm of the universal formal group law. Running the calculations gives

$$\log_\phi(x) = \sum \frac{\phi([\mathbb{C}P^{2n}])}{2n+1} x^{2n}.$$

Dually, we have the characteristic series associated with ϕ. If $\exp_\phi$ denotes the power series which is inverse to $\log_\phi$, then we have the characteristic series associated to ϕ, given by

$$K_\phi(x) = \frac{x}{\exp_\phi(x)}.$$

The relevance of such notion is made clear by observing that the product of the evaluations of K_ϕ in Chern roots gives a characteristic class.

Example 3. *The Todd genus is the genus whose characteristic class is given by*

$$K_\phi(x) = \frac{x}{1-e^{-x}}.$$

For a vector bundle P with Chern roots α_i, the Todd class of P is the product

$$\mathrm{Td}(P) = \prod_i K_\phi(\alpha_i).$$

Example 4. *The $\hat{A}$ genus is the genus whose characteristic series is given by*

$$K_\phi(x) = \frac{x/2}{\sinh(x/2)} = \frac{x}{e^{x/2} - e^{-x/2}}.$$

Different genera may behave differently with respect to specific kinds of fiber bundles. In particular, if $P \to M$ is a fiber bundle with fiber V, then the formula

$$\phi(P) = \phi(M)\phi(V) \tag{1}$$

whenever

- ϕ is the $\hat{A}$ genus and F is a spin manifold;
- or ϕ is the signature genus and the action of $\pi_1(M)$ on $H^*(V)$ is trivial ([CHS57]).

Ochanine and Taubes classified the genera for which Equation 1 holds when both conditions above are simultaneously satisfied. This is expressed in the following theorem.

Theorem 6. *Given ϕ a $\mathbb{C}$-valued genus. Then*

$$\phi(P) = \phi(M)\phi(V)$$

holds for any fiber bundle $V \to P \to M$ where V is a spin manifold and the fundamental group of M acts trivially on V if and only if the logarithm series associated with ϕ is an elliptic integral

$$\log_\phi(x) = \int_0^x \frac{1}{\sqrt{1 - 2\delta t^2 - \epsilon t^4}} dt. \tag{2}$$

*In such case, ϕ is said to be an **elliptic genus**.*

The signature and A-hat genus can be seen as degenerate cases of elliptic genera. They correspond precisely to the pairs of values (δ, ϵ) that make the discriminant of the polynomial inside the square root in Equation 2 to be zero.

An interesting fact about genera is that they are all evaluations of characteristic classes on fundamental classes. By the splitting principle, to define a characteristic class, it is enough to do so in line bundles. So if K_ϕ is the characteristic series of the genus ϕ, we let $k_\phi(L) = K_\phi(c_1(L))$.

Theorem 7. *Given a genus ϕ and its characteristic class k_ϕ, we have*

$$\phi(M) = \langle k_\phi(\mathrm{T}\, M), [M] \rangle$$

The Theorem 7 above will be extremely important later, as the right hand side is included in Atiyah-Singer Index theorem.

We finish this section with some quick words concerning the universal elliptic genus. A "formula" for the characteristics series of the universal elliptic genus is well known. For the braves who want to take a look at that, it is

$$K_\phi(x) = \frac{x/2}{\sinh(x/2)} \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})},$$

where q above is related with a certain modular form built over Equation 2.

It may be really hard to guess what kind of properties to expect from the universal genus straight from its formula. Thankfully, we have another approach available, based on some insights by Witten in [Wit88].

4 Dirac Operators in Loop spaces

For a moment, let us discuss the index of certain Dirac operators in loop spaces. The reason why we will do that is because it is possible, considering the index of such operators, to build a universal elliptic genus, which is called the *Witten genus*, after (unsurprisingly) Witten. The next theorem stablishes when does $\mathcal{L}M$, the free loop space of M, has a spinor bundle, which will be useful to define a Dirac operator on $\mathcal{L}M$.

Theorem 8. *Given a manifold M, $\mathcal{L}M$ has a spinor bundle if and only if the first Pontryagin class $p_1(M)$ of M vanishes.*

Under the conditions of this theorem, we can build a Dirac operator on $\mathcal{L}M$, although the construction is complicated. The basic idea is to use the embedding of M into $\mathcal{L}M$ via the constant loops to split the tangent bundle $\mathrm{T}\,\mathcal{L}M = \mathrm{T}\,M \oplus N_M$ into components corresponding to the tangent bundle of M and its normal bundle. Then, one can use the spin structure on M to obtain a Dirac operator corresponding to the first component. The really involving part is building the Dirac operator corresponding to the second component, which needs a great deal of representation theory of Lie superalgebras. After conveniently tensoring these Dirac operators, we obtain an honest Diract operator

$$\mathcal{D}^M : \Gamma(S^+) \to \Gamma(S^-)$$

over the spinor bundle in $\mathcal{L}M$. Furthermore, this Dirac operator commutes with the S^1 action on $\mathcal{L}M$ given by translation along a loop. Pragmatically, that means we can do "things" S^1-equivariantly. The "things" we will be particularly interested is applying the Atyiah-Singer Index Theorem.

A modification on the construction, namely, tensorizing with another bundle $P \to \mathcal{L}M$ yields a twisted Dirac operator

$$\mathcal{D}^M_P : \Gamma(S^+ \otimes P) \to \Gamma(S^- \otimes P).$$

The interesting case for us is the case where $P = S$, so $\mathcal{D}^M_S$ is an S^1-equivariant twisted Dirac operator on $\mathcal{L}M$ and we may take its S^1-index.

Remark. *In full generality, one doesn't need a spin bundle S over the whole $\mathcal{L}M$, but only the existence of the a product bundle $S \otimes S$ with a grading $S \otimes S^\pm$. The construction of $\mathcal{D}^M_P$ has some extra caveats in this case.*

Applying the S^1-equivariant Index Theorem for $\mathcal{D}^M_S$ and running the calculations on the topological side of the theorem connects the index of $\mathcal{D}^M_S$ to the right hand side of the equation present in Theorem 7 for some elliptic genus. A closer inspection of the formulas makes clear that this elliptic genus is the universal one. In more precise terms, we have the theorem below.

Theorem 9. *If ϕ denotes the universal elliptic genus, then*

$$\phi(M) = C \cdot \mathrm{Ind}_{\mathrm{S}^1} \mathcal{D}^M_S.$$

where C is a "coherence" constant coming up from the fact that elliptic genera form modular forms.

This is a surprising result: the universal elliptic genus, up to a rescaling, is given by the indices of Dirac operators on loop spaces. This fact helps to explain a lot of observed qualitative behaviors of the universal elliptic genus, like a certain similarity with what is expected from characters of representations of $\mathrm{Diff}(S^1)$. Witten's paper ([Wit88]) is basically a suggestion of the procedure to define our relevant Dirac operators on $\mathcal{L}M$ with a proposal of an explanation of why certain properties of the universal elliptic genus, as the one we just commented about, hold.

The explanations are mostly heuristic, though, and the general line is to lift the interesting properties from structures that exist in $\mathcal{L}M$ to things happening in the genus. More specific examples of such "lifts" comprehend the consequences of the existence of a $\mathrm{Diff}(S^1)$ action on $\mathcal{L}M$ given by reparametrization. The passage to the elliptic genus world involves some constructions using conformal quantum field theory and classic complex analytical results. Due to this work, the value expressed in the equation present in Theorem 9 is called the **Witten genus**.

5 Elliptic cohomology today

Elliptic cohomology theories are related with many different areas of mathematics. Nowadays, we have two strong trends related with elliptic cohomology lying in string theory and derived algebraic geometry.

Maybe the most outstanding interest in string theory is the unsolved *Stolz Conjecture*. The Stolz conjecture says that if a string manifold has Ricci positive curvature, then its Witten genus is trivial.

Another related hot topic is the cohomology theory known as *tmf*, which stands for topological modular forms. This is, in some sense, the universal elliptic cohomology theory, with the license of not being an actual elliptic cohomology theory. As we have a universal elliptic genus, we have a universal elliptic cohomology, and one "corresponds" to the other. The issue is that *tmf* is not really an elliptic cohomology theory: there is no elliptic curve associated. The reason, or a reason, is that *tmf* is built as a certain colimit over all the elliptic curves, organized in the *moduli stack of elliptic curves*. However, this colimit does not live anymore in the moduli stack of elliptic curves, but instead in its Deligne-Mumford compactification.

Physically, we have a universal spin K-orientation known as **Atiyah-Bott-Shapiro orientation**. A spin K-orientation here means a homomorphism from the spin bordism ring MSpin to the K-theory ring. The genus associated with this orientation is the A-hat genus. When one goes up from spin structures to string structures, one can ask what happens with this universal K-orientation, or more precisely, what is the string-analogous of this *universal orientation - universal genus* relation.

The answer to that question is to change the domain of the homomorphism from MSpin to MString and replace K theory by *tmf*. In this case, instead of the A-hat genus, we end up with the Witten genus.

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