A crash introduction to the Conley index

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The *Conley index* is a topological tool for studying dynamical systems. It allows us to analyze flows in topological spaces, gathering information about their invariant sets. This information will be given in terms of (co)homology and homotopy. This text aims to give a very basic introduction to the Conley index theory. We are going to build the index and look at some of its most interesting or useful properties.

The core concept of continuous dynamical systems is the *flow*. It is defined as follows.

Definition 1. A **flow** in a topological space X is a continuous function $\phi : \mathbb{R} \times X \to X$ satisfying the following properties.

- $\phi(0, -)$ is the identity in X;
- $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $x \in X$ and for all $t \in \mathbb{R}$.

We now can define the **orbit** of a point x under a flow ϕ to be the set $\phi(\mathbb{R}, x)$. Additionally, we say that a subset $S \subset X$ is an **invariant set** if $\phi(\mathbb{R}, S) = S$. Generally, it is not easy to study dynamical systems just by manipulating the flow that defines it, so we go for a more qualitative approach. Invariant sets are one of the interesting things in a dynamic system we would like to know more about. The index is useful to study such sets, specifically when talking about *isolated invariant sets*, whose definition is given below.

Definition 2. An **isolated invariant set** (IIS) is an invariant set S that admits a compact neighborhood N such that

$$S = \mathrm{Inv(N, \phi) := \{x \in N : \phi(\mathbb{R}, x) \subset N\} \subset Int(N).}$$

In this case, we call N an **isolating neighborhood** for S.

This definition captures the setup where no orbit that touches the boundary of N is contained in N, that is, the orbit must leave N in the future or in the past. It may return to N after leaving, but must spend some time out of N. Intuitively, the isolating neighborhood separates the invariant set from the rest of the dynamical system as objects outside the isolating neighborhood "don't interact" with the IIS. Indeed, we further will be interested in the internal dynamics of the isolating invariant set and we will see we don't need outside data for studying it.

Let us look at some examples of isolated invariant sets.

Example 3. Consider the flow $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\phi(t, x) = x \cdot e^t$. One can check it is, indeed, a flow. In addition, the point $x = 0$ is a fixed point. A quick analysis shows that if $x \neq 0$, the orbit of x will move away from 0. We can synthesize this dynamical system by the following portrait.

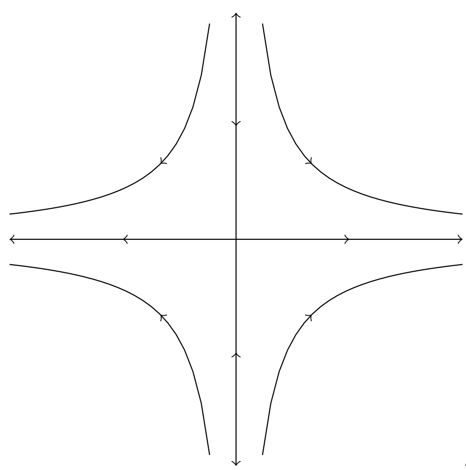
In this case, $\{0\}$ is an isolated invariant set, with any of its compact neighborhoods being admissible isolating neighborhoods.

Example 4. A more interesting (and a classical) example is the flow $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\phi(t, (x, y)) = (x \cdot e^t, y \cdot e^{-t})$.

This flow describes the solutions for the system of differential equations

$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

We can draw the following portrait to represent the system:



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The origin is an isolated invariant set, isolated by any of its compact neighborhoods.

It is an upgrade of the previous example, but a new feature starts to appear: hyperbolicity. A great part of our interest in this flow is due to the fact that $(0, 0)$ is a hyperbolic point, one of the simplest examples we can build.

The definition of the Conley index relies on the concept of an *index pair*, whose definition follows below.

Definition 5. If S is an IIS in a flow ϕ, an index pair for S is a pair of compact sets (N_1, N_2), with $N_2 \subset N_1$ such that

1. $S = \text{Inv}(\overline{N_1 \setminus N_2})$ and $N_1 \setminus N_2$ is a neighborhood of S;
2. if $x \in N_2$, then $\phi([0,t],x) \subset N_1 \Rightarrow \phi([0,t],x) \subset N_2$;
3. for all $x \in N_1$ and for all $t > 0$, if $\phi(t,x) \notin N_1$, then there is $t' \in [0,t]$ such that $\phi(t',x) \in N_2$ and $\phi([0,t'],x) \subset N_1$.

In general terms, the first property state that $N_1 \setminus N_2$ is an isolating neighborhood for S, while the last two properties say that a point in N whose future orbit leaves N must pass by L first, and will not leave L unless it leaves N at the same time.

Now we have the following theorem due to Conley.

Theorem 6 (Conley 1976)**.** *Every isolated invariant set has an index pair.*

With everything set up right now, we can define the Conley index.

Definition 7. Given an index pair (N_1, N_2) for an IIS S in a flow ϕ, we define the **homotopy Conley index** of S as

$$h(S, \phi) = [(N_1/N_2), [N_2]],$$

the homotopy type of the pointed space $((N/L), [L])$.

We also define the **homology Conley index** of S as

$$CH_\bullet(S, \phi) = H_\bullet(N_1/N_2, [N_2]).$$

We will often omit the flow ϕ when it is clear what it must be.

There is also the cohomological index, defined by replacing homology with cohomology. Conley shows that the index is independent of the choice of the index pair, so that it is well-defined.

The philosophy of the Conley index is that invariant sets are very hard to be studied directly. So we use isolating neighborhoods to do so. The usefulness of the index is given by its properties, which we will explore now.

First, if $S = \emptyset$, then $(\emptyset, \emptyset)$ forms an index pair for S. Hence, if $S = \emptyset$, the index is trivial. The contrapositive gives us the ability to detect invariant sets.

Theorem 8. *If $h(S) \neq 0$, then $S \neq \emptyset$.*

The same holds for the homological or cohomological index. We also a homotopy invariance of the index, in the following sense.

Theorem 9. *Let $\{\phi_\lambda\}, \lambda \in [0,1]$ be a family of flows varying continuously in λ (a homotopy between ϕ_0 and ϕ_1). Fix some compact set N and denote $S_\lambda := \text{Inv}(\text{N}, \phi_\lambda)$. If N is an isolating neighborhood for all S_λ, then all $h(S_\lambda, \phi_\lambda)$ coincide.*

This last property is very useful since it allows us to transform complicated flows into homotopical simple ones which usually are way easier to study. The typical use of the Conley index in the studies of concrete dynamical systems is based on these two first properties. One asks if there is some invariant set inside a region and then replace the original flow with a simpler one, where it is easier to verify the existence of a non-empty invariant set in a relevant neighborhood. Then, we can conclude that indeed there was an invariant set in the region we were interested in.

For illustration purposes, we show how the Conley index indicates the existence of invariant sets in examples 3 and 4.

In the first example, we can see that the boundary of any interval $[-a, a]$, with $a > 0$, doesn't intersect with invariant sets contained in $[-a, a]$, since $\phi(t, -a) < -a$ and $\phi(t, a) > a$ for all $t > 0$, which means the orbits of the boundary points will leave $[-a, a]$. So $N = [-a, a]$ is an isolating neighborhood of some set, which could be empty.

We select the index pair (N, L), where

$$\begin{aligned} N &= [-a, a] \\ L &= \{-a, a\} \end{aligned}$$

It should not be hard to convince yourself that this is an index pair. Now, we take the quotient N/L, which is homeomorphic to the circle S^1. Since S^1 has a non-trivial homotopy type, we conclude that N contains a non-empty isolated invariant set.

The work is analogous in the second example, where we have a hyperbolic point. This time, our index pair will be (N, L) with

$$\begin{aligned} N &= [-a, a] \times [-a, a] \\ L &= (\{-a\} \times [-a, a]) \cup (\{a\} \times [-a, a]). \end{aligned}$$

In this case, the quotient N/L will give us a cylinder, which also doesn't have a trivial homotopy type. In general, this example can be generalized to hyperbolic points with any number as the dimension of its unstable manifold. In fact, we have the following theorem.

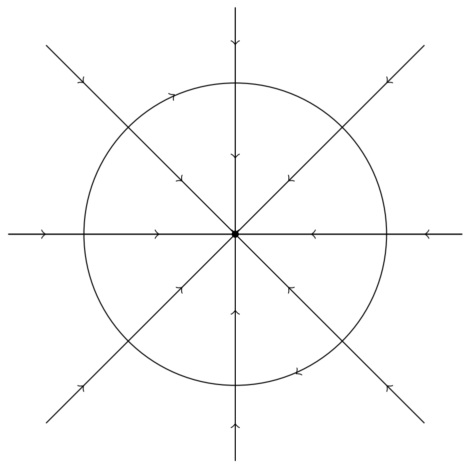
Theorem 10. *Let x be a hyperbolic fixed point in the flow ϕ. If the unstable manifold of x has dimension n, then*

$$CH_k(\{x\}, \phi) = \begin{cases} \mathbb{Z} \text{ if } k = n \text{ or } k = 0 \\ 0 \text{ otherwise.} \end{cases} .$$

In other words, $\{x\}$ has the homology Conley index equal to the homology of the n-sphere. This theorem follows from a linearization of the dynamical system where x is hyperbolic and the application of the procedure done previously in two dimensions, with minor adaptations.

Now we present a last result that helps us to understand the internal dynamics of an isolated invariant set. What it means is that inside an isolated invariant set, there can be many interesting things going on. We could have other invariant sets and orbits connecting them.

Example 11. For a concrete example, consider a dynamic system in the plane with a fixed point at the origin and a periodic orbit which is a circle centered around this point. All the other orbits flow directed to the center. The orbits outside the circle approach it asymptotically while the ones inside it, approach the central fixed point. This system is shown in the following portrait.



We shall observe that the region enclosed by the circle (including itself) is an invariant set. Not only that, but it is an isolated invariant set: a disk of greater radius is an admissible isolated neighborhood. We could look inside the invariant disk and we will find out that it is composed of a union of three other invariant sets: the circular periodic orbit, the center fixed point, and the orbits that go from the periodic orbit to the fixed point.

We can go even further and check that both the fixed point and the periodic orbit are isolated invariant sets. They are IISs connected by orbits that are contained in the invariant disk. We start to notice that invariant sets can have lifeful internal dynamics.

In this particular case, we have our invariant disk S being equal to the union $S = \{x_0\} \cup M \cup C(M, x_0)$, where M is the circular periodic orbit and $C(M, x_0)$ is the set of points in the connecting orbits from M to x_0, that is, the points on trajectories that start in M and go to x_0 (asymptotically). Both $\{x_0\}$ and M are IIS, so let us calculate the Conley index of them.

For the periodic orbit, will N_1 can be an annulus that contains the orbit and N_2 the inner boundary of the annulus. The quotient N_1/N_2 is homeomorphic to a disk and thus has trivial homology groups.

The story is not different for the fixed point. N_1 can be a small disk centered around x_0 and N_2 will be the empty set. Then, the quotient N_1/N_2 is again homeomorphic to a disk and has trivial homology groups.

What about the Conley index of the whole invariant disk? N_1 can be a bigger disk containing S and N_2 will be empty again. So in this case, $CH_\bullet(S) \cong CH_\bullet(x_0) \oplus CH_\bullet(M)$.

We may ask if this is always true. Unfortunately (or fortunately) it is not always true. But we know a sufficient condition for it to happen.

Theorem 12. *Let S be an IIS such that*

$$S = \bigcup_{i=1}^{n} M_i$$

where each M_i is an IIS, then

$$CH_\bullet(S) = \bigoplus_{i=1}^{n} CH_\bullet(M_i).$$

At first glance, we could be sad about the failure of the other direction of this theorem. But this failure actually comes with its own advantages. In summary, we may have a couple of IISs and we want to know if there are orbits connecting them. Take an IIS that contains all the previous isolated invariant sets and compare the Conley indices. If they don't match, we know for sure that there exists connecting orbits between our IISs. But if the indices match, we can not conclude anything, as the example 11 shows.

There are still many topics of the Conley index theory to cover. We did not say a word about the index for discrete dynamical systems, which is slightly more delicate since some of the basic theorems we used for defining the index are not available.

Other interesting topics include how the Conley index can lead to homological chains related to decompositions of isolated invariant sets. There is also a strong relation between the Conley index and Morse theory, where the Conley index can be used to prove, for example, the Poincaré-Hopf theorem. The Conley index further expands or leads to other theories, like the $\mathscr{L}\mathscr{S}$-index and Floer homology.

References

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